Two-Dimensional Spatio-Temporal Dynamics of Analog Image Processing Neural Networks

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***Abstract*— A typical analog image-processing neural network consists of a two-dimensional (2-D) array of simple processing elements. When it is implemented with CMOS LSI, two dynamics issues naturally arise:**

1) **Parasitic capacitors of MOS transistors induce temporal dynamics. Since a processed image is given as the stable equilibrium point of temporal dynamics, a temporally unstable chip is unusable.**
2) **Because of the array structure, the node voltage distribution induces spatial dynamics, and the node voltage distribution could behave in a wild manner, e.g., oscillatory, which is undesirable for image-processing purposes.**

A discussion of these issues for one-dimensional cases is found in [1]. This paper extends its results to 2-D cases and also derives several explicit formulas and relationships for the 2-D dynamics, which are useful for the design and analysis of the class of networks of interest. Specifically, the following are derived: i) explicit spatial and temporal stability conditions and their equivalency, ii) spatial impulse responses, iii) spatial frequency responses, iv) power consumption, v) time constants, vi) relationships between spatial frequency responses and stability, vii) relationships between power consumption and stability, viii) relationships between spatial impulse responses and the discrete Fourier transform of network parameters, ix) relationships between spatial impulse responses and the inverse Z-transform of a transfer function, x) relationships between spatial frequency responses and time constants, xi) relationships between spatial frequency responses and equivalent circuits, xii) the characteristics of stable and unstable network dynamics, and xiii) hexagonal as well as square grid network dynamics.

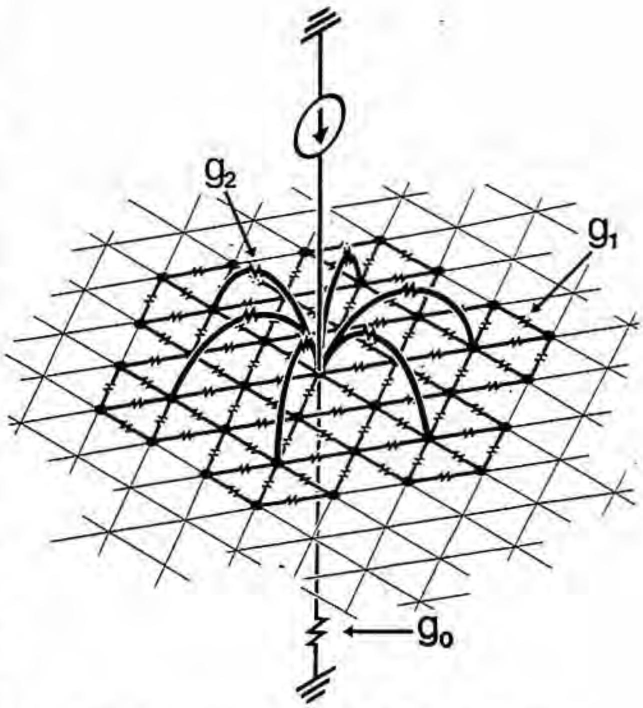

Fig. 1. The image-smoothing neuro chip. Only one unit is shown.

I. Introduction

THIS study has been motivated by spatial versus temporal stability issues of analog image-processing neuro chips. The image-smoothing neuro chip [2] implemented by one of the authors, for instance, consists of a regular array of photosensors with conductances $g_0 > 0, g_1 > 0, g_2 < 0$ (Fig. 1). We refer the reader to [2] for the chip details. Since the chip involves negative conductances g_2, both spatial and temporal stability issues naturally arise. There are two intriguing elements. First, our earlier numerical experiments suggested that generally a neuro chip is temporally stable if and only if it is spatially stable, where spatial stability means that the node voltage distribution behaves "properly." Second, spatial dynamics naturally induces a discrete linear dynamical system so that its stability should be checked by its eigenvalues. "A discrete linear dynamical system is stable if and only if all the eigenvalues lie inside the unit circle of the complex plane." This statement turned out to be false. Namely, due to the noncausal nature of the dynamics, if λ is an eigenvalue, so is $1/\lambda$, and hence the stability condition for causal linear systems is never satisfied.

Most of the fundamental issues involving these two elements have been settled in [1] for one-dimensional (1-D) array cases. For instance, a network is temporally stable if and only if it is spatially stable, except for a set of Lebesgue measure zero in the parameter space. Another fundamental result was that a network is spatially stable if and only if the eigenvalues of the dynamics are off the unit circle, even though they can be outside the unit circle. These results are far from trivial. One of the reasons that makes these results crucial is the boundary conditions associated with the finiteness of a network. Even if the eigenvalue conditions are satisfied, solutions can oscillate or explode if the boundary conditions are inappropriate.

Although our results in [1] are completely rigorous, the results are for 1-D cases only. The purpose of this paper is two-fold:

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1) First, the stability results obtained in [1] are extended rigorously to two-dimensional 2-D cases, for which several new ideas are necessary.
2) Second, this paper discusses the 2-D dynamics issues and derives the following explicit formulas and relationships which are useful for designing and analyzing the class of filters described: i) spatial impulse responses, ii) spatial frequency responses, iii) power consumption, iv) time constants, v) relationships between spatial frequency responses and stability, vi) relationships between power consumption and stability, vii) relationships between spatial impulse responses and the discrete Fourier transform of network parameters, viii) relationships between spatial impulse responses and the inverse Z-transform of a transfer function, ix) relationships between spatial frequency responses and time constants, x) relationships between spatial frequency responses and equivalent circuits, xi) the characteristics of stable and unstable network dynamics, and xii) hexagonal as well as square grid network dynamics.

Related Works: The paper by White and Wilson [3] is vitally related to the present one. Some of the results in [3] regarding stability are similar to ours [1) mentioned above], even though the two approaches are completely different and there are several different issues addressed; e.g., [3] discusses the relationship between boundary conditions and temporal stability while [1] and this paper discuss the relationship between boundary conditions and spatial stability. In addition, this paper describes not only stability issues but also dynamics issues, and our approach leads to several explicit results [2) mentioned above].

Our approach in this paper is a systematic exploitation of the block circulant network structure for 2-D cases and the circulant network structure for 1-D cases. Speaking roughly, a block circulant network has a "two-torus" (doughnut) structure while a circulant network has a "ring" structure. Precise definitions will be given later. This exploitation of the block circulant and circulant structures has been used in several other previous works, e.g., [4]–[6]. In the present paper, however, since actual chips are not block circulant, our results for block circulant and circulant networks would be of little value unless block circulant and circulant networks behave in a manner similar to the networks that are uniform block and uniform band, respectively (see Section II-B for a precise definition). Also the networks include active elements (negative conductances) and thus the boundary conditions are crucial as already shown in [1] for 1-D cases. Therefore, a careful examination must be done to determine whether this approach can be of any use. It is shown that as the network size grows, responses of stable block circulant and circulant networks behave in a manner similar to those of stable uniform block and uniform band networks, respectively, which makes the approach valid.

We also remark that these theoretical results have important and practical consequences for the design and implementation of the class of filters with the analog CMOS LSI. Another example of the stability study for the analog early vision chip is the work of Standley and Wyatt [8] who investigated the stability conditions of Mead's chips [7] because they could be unstable in certain conditions. Another chip [2] which motivated the present study has negative conductances and thus it can easily be unstable. If the analog network is unstable, it is unusable, and thus these stability issues are important in analog neuro chips as well as in many other analog circuits.

II. FORMULATION

A. Circulant Networks

Let us first explain circulant networks before explaining block circulant networks. Consider a 1-D network with N nodes numbered zero through $N-1$, where each node k is connected with a current source u_k, a (possibly negative) conductance g_0, a capacitance c_0 to ground, and (possibly negative) conductances g_l and capacitances c_l to the lth nearest neighbor nodes ($l = 1, 2, \cdots, m$). We neglect inductance components here because most analog image-processing neuro chips are implemented by CMOS [2], [7], where inductances are practically zero. The network is called circulant if the rightmost and leftmost nodes are connected (ring-shaped). Fig. 2 shows a circulant network where $m = 2$. Observe that the Kirchoff current law (KCL) at node k reads

$$\begin{aligned}&\frac{d}{dt}\{(c_0+2c_1+2c_2)v_k - c_1(v_{k+1}+v_{k-1})\\&\qquad - c_2(v_{k+2}+v_{k-2})\}\\&\quad = -(g_0+2g_1+2g_2)v_k + g_1(v_{k+1}+v_{k-1})\\&\qquad + g_2(v_{k+2}+v_{k-2}) + u_k.\end{aligned}$$

Since the network is circulant, node zero is connected to node $N-1$ as a nearest neighbor node and hence the state equation of the network where $m=2$ is given by

$$\boldsymbol{B}\frac{d}{dt}\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v} + \boldsymbol{u}$$

where

$$\begin{aligned}&\boldsymbol{v} := (v_0, v_1, \cdots, v_{N-1})^T \in \mathcal{R}^N,\\&\boldsymbol{u} := (u_0, u_1, \cdots, u_{N-1})^T \in \mathcal{R}^N,\\&\boldsymbol{A} := \{A(i,j)\} \in R^{N\times N}, \qquad i,j = 0,1,\cdots,N-1,\\&A(i,j) := \begin{cases} a_0 & \text{when } i=j\\ a_1 & \text{when } (i-j) \bmod_N = \pm 1\\ a_2 & \text{when } (i-j) \bmod_N = \pm 2\\ 0 & \text{otherwise}\end{cases}\\&a_0 := -g_0 - 2(g_1+g_2), \quad a_1 := g_1, \quad a_2 := g_2,\\&\boldsymbol{B} := \{B(i,j)\} \in R^{N\times N}, \qquad i,j = 0,1,\cdots,N-1,\\&B(i,j) := \begin{cases} b_0 & \text{when } i=j\\ b_1 & \text{when } (i-j) \bmod_N = \pm 1\\ b_2 & \text{when } (i-j) \bmod_N = \pm 2\\ 0 & \text{otherwise}\end{cases}\\&b_0 := c_0 + 2(c_1+c_2), \quad b_1 := -c_1, \quad b_2 := -c_2.\end{aligned}$$

Thus, for a general m, matrixes $\boldsymbol{A}$ and $\boldsymbol{B}$ are of the forms

$$\begin{aligned}&\boldsymbol{A} := \{A(i,j)\} \in R^{N\times N}, \qquad i,j = 0,1,\cdots,N-1,\\&A(i,j) := \begin{cases} a_k & \text{when } (i-j) \bmod_N = \pm k, \quad k = 0,\cdots,m\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

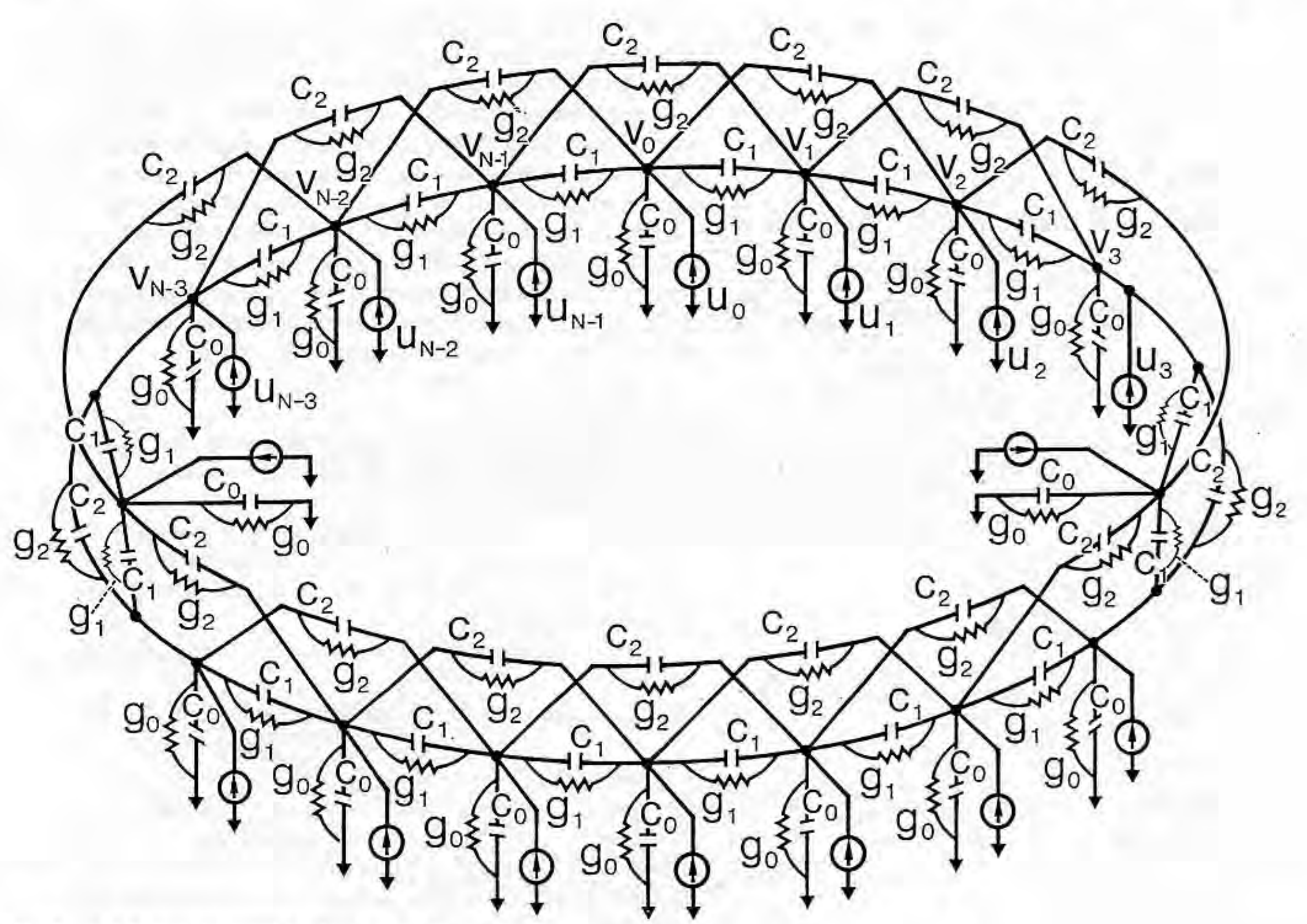


Fig. 2. A 1-D circulant network where $m = 2$.

$$a_0 := -g_0 - 2\sum_{p=1}^{m} g_p, \quad a_p := g_p \, (p = 1, 2, \cdots, m) \quad (2.1)$$

$$\boldsymbol{B} := \{B(i,j)\} \in R^{N \times N}, \qquad i, j = 0, 1, \cdots, N-1,$$

$$B(i,j) := \begin{cases} b_k & \text{when } (i-j) \bmod_N = \pm k, \quad k = 0, \cdots, m \\ 0 & \text{otherwise} \end{cases}$$

$$b_0 := c_0 + 2\sum_{p=1}^{m} c_p, \quad b_p := -c_p \, (p = 1, 2, \cdots, m). \quad (2.2)$$

The name "circulant network" comes from the fact that matrixes of the forms (2.1) and (2.2) are called (symmetric) circulant [9]. If nodes zero and $N-1$ are disconnected and the disconnected resistors are connected to ground [3], (2.1) and (2.2) should be replaced by

$$\boldsymbol{A} := \{A(i,j)\} \in R^{N \times N}, \qquad i, j = 0, 1, \cdots, N-1,$$

$$A(i,j) := \begin{cases} a_k & \text{when } i - j = \pm k, \quad k = 0, \cdots, m \\ 0 & \text{otherwise} \end{cases}$$

$$\boldsymbol{B} := \{B(i,j)\} \in R^{N \times N}, \qquad i, j = 0, 1, \cdots, N-1,$$

$$B(i,j) := \begin{cases} b_k & \text{when } i - j = \pm k, \quad k = 0, \cdots, m \\ 0 & \text{otherwise} \end{cases}$$

which are uniform band matrixes, and the corresponding network is called a uniform band network. The networks discussed in [1], [3], and [10] are of this type.

B. Block Circulant Networks

Now let us consider a 2-D network with $N_1 \times N_2$ nodes numbered (0, 0) through $(N_1 - 1, N_2 - 1)$, where each node (k_1, k_2) is excited by a current source u_{k_1,k_2} and has a (possibly negative) conductance g_0 and a capacitance c_0 to ground, and (possibly negative) conductances g_{l_1,l_2} and capacitances c_{l_1,l_2} to nodes $(k_1 + l_1, k_2 + l_2)$ for $l_1 = 0, \pm 1, \pm 2, \cdots \pm m_1, l_2 = 0, \pm 1, \pm 2, \cdots \pm m_2$ except for $l_1 = l_2 = 0$. Note that $g_{l_1,l_2} = g_{-l_1,-l_2}$ and $c_{l_1,l_2} = c_{-l_1,-l_2}$ because node (k_1, k_2) connects to node $(k_1 + l_1, k_2 + l_2)$ with g_{l_1,l_2} and c_{l_1,l_2} whereas node $(k_1 + l_1, k_2 + l_2)$ connects to node $((k_1 + l_1) - l_1, (k_2 + l_2) - l_2))$, i.e., node (k_1, k_2) with $g_{-l_1,-l_2}$ and $c_{-l_1,-l_2}$ and hence $g_{l_1,l_2} = g_{-l_1,-l_2}$ and $c_{l_1,l_2} = c_{-l_1,-l_2}$. The network is said to be block circulant if the rightmost and leftmost nodes are connected together and the top and bottom nodes are connected together, and thus the network is of a torus structure. Fig. 3 shows a block circulant network where $m_1 = m_2 = 1$. Since the KCL equation at node (k_1, k_2) reads

$$\begin{aligned} \frac{d}{dt}\{&(c_0 + 2c_{1,1} + 2c_{1,0} + 2c_{1,-1} + 2c_{0,1})v_{k_1,k_2} \\ &- c_{1,1}(v_{k_1+1,k_2+1} + v_{k_1-1,k_2-1}) \\ &- c_{1,0}(v_{k_1+1,k_2} + v_{k_1-1,k_2}) \\ &- c_{1,-1}(v_{k_1+1,k_2-1} + v_{k_1-1,k_2+1}) \\ &- c_{0,1}(v_{k_1,k_2+1} + v_{k_1,k_2-1})\} \end{aligned}$$

$$\begin{aligned}= &-(g_0+2g_{1,1}+2g_{1,0}+2g_{1,-1}+2g_{0,1})v_{k_1,k_2}\\ &+g_{1,1}(v_{k_1+1,k_2+1}+v_{k_1-1,k_2-1})\\ &+g_{1,0}(v_{k_1+1,k_2}+v_{k_1-1,k_2})\\ &+g_{1,-1}(v_{k_1+1,k_2-1}+v_{k_1-1,k_2+1})\\ &+g_{0,1}(v_{k_1,k_2+1}+v_{k_1,k_2-1})+u_{k_1,k_2}\end{aligned}$$

so that the state equation of the network for general m_1 and m_2 is given by

$$B_b\frac{d}{dt}v_b = A_b v_b + u_b \tag{2.3}$$

where

$$\begin{aligned}
&v_b := \{v'_k\}^T \in \mathcal{R}^{N_1N_2}, \quad u_b := \{u'_k\}^T \in \mathcal{R}^{N_1N_2},\\
&\qquad k = 0,1,\cdots,N_1N_2-1,\\
&v'_k := v_{k_1,k_2}, \quad u'_k := u_{k_1,k_2}, \quad k := k_1N_2 + k_2,\\
&\qquad k_1 = 0,1,\cdots,N_1-1,\\
&\qquad k_2 = 0,1,\cdots,N_2-1,\\
&A_b := \{A(i,j)\} \in \mathcal{R}^{N_1N_2\times N_1N_2},\\
&\qquad i,j = 0,1,\cdots,N_1-1,\\
&A(i,j) := \begin{cases} A_l & \text{when } (i-j)\ \mathrm{mod}_{N_1} = l,\\ & \quad l = 0,\pm1,\cdots,\pm m_1 \quad \in R^{N_2\times N_2}\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

and for $l = 0,\pm1,\pm2,\cdots,\pm m_1$

$$\begin{aligned}
&A_l := \{A_l(i,j)\} \in \mathcal{R}^{N_2\times N_2}, \qquad i,j = 0,1,\cdots,N_2-1,\\
&A_l := \begin{cases} a_{l,k} & \text{when } (i-j)\ \mathrm{mod}_{N_2} = k,\\ & \quad k = 0,\pm1,\cdots,\pm m_2\\ 0 & \text{otherwise}\end{cases}\\
&a_{0,0} := -\sum_{p=-m_1}^{m_1}\sum_{q=-m_2}^{m_2} g_{p,q}, \qquad g_{0,0} := g_0,\\
&a_{p,q} := g_{p,q}, \quad a_{p,q} = a_{-p,-q}, \quad (p = 0,\pm1,\pm2,\cdots,\pm m_1;\\
&\qquad q = 0,\pm1,\pm2,\cdots,\pm m_2; \quad (p,q)\neq(0,0)),\\
&B_b := \{B(i,j)\} \in \mathcal{R}^{N_1N_2\times N_1N_2},\\
&\qquad i,j = 0,1,\cdots,N_1-1,\\
&B(i,j) := \begin{cases} B_l & \text{when } (i-j)\ \mathrm{mod}_{N_1} = l,\\ & \quad l = 0,\pm1,\cdots,\pm m_1 \quad \in R^{N_2\times N_2}\\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{2.4}$$

and for $l = 0,\pm1,\pm2,\cdots,\pm m_1$

$$\begin{aligned}
&B_l := \{B_l(i,j)\} \in \mathcal{R}^{N_2\times N_2},\\
&\qquad i,j = 0,1,\cdots,N_2-1,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&B_l(i,j) := \begin{cases} b_{l,k} & \text{when } (i-j)\ \mathrm{mod}_{N_2} = k,\\ & \quad k = 0,\pm1,\cdots,\pm m_2\\ 0 & \text{otherwise}\end{cases}\\
&b_{0,0} := \sum_{p=-m_1}^{m_1}\sum_{q=-m_2}^{m_2} c_{p,q}, \qquad c_{0,0} := c_0,\\
&b_{p,q} := -c_{p,q}, \quad b_{p,q} = b_{-p,-q}, \quad (p = 0,\pm1,\pm2,\cdots,\\
&\qquad \pm m_1; \quad q = 0,\pm1,\pm2,\cdots,\pm m_2;\\
&\qquad (p,q)\neq(0,0)).
\end{aligned} \tag{2.6}$$

Matrixes A_b and B_b are called (symmetric) block circulant with circulant blocks [9].

Circulant and block circulant matrixes enjoy many interesting properties [9] and we will fully exploit these properties. Note that a network being circulant or block circulant is equivalent to its boundary conditions being periodic. We would like to emphasize that boundary conditions are critical for the class of problems we are discussing. Namely, even if the eigenvalue conditions are satisfied for the spatial dynamics, inappropriate boundary conditions may force the solution to blow up (see [1]). It is extremely difficult, if not impossible, to obtain analytical conditions for stability as well as other properties for uniform block networks, while several interesting analytical results can be obtained for block circulant networks.

On the other hand, when A_l and B_l $(l = 0,\pm1,\cdots,\pm m_1)$ are uniform band matrixes and they are assigned in A_b and B_b in uniform bands, respectively, matrixes A_b and B_b are called uniform block and the corresponding network is said to be a uniform block network.

We further remark that all the results for 2-D networks described in this paper are also valid in 1-D cases by specifying $N_2 = 1$ and $m_2 = 0$ and the results are consistent with those for 1-D networks in [1].

Standing Assumptions: We will hereafter assume the following:

i) Capacitance matrix B_b is positive definite for all N_1, N_2, and

ii) $a_{0,0} < 0$.

Assumption i) is very mild because, for example, if all capacitances $c_0, c_{0,1}, \cdots c_{m_1,m_2}$ are positive, B_b is positive definite. The reason for assuming positive definiteness of B_b for all N_1 and N_2 is to derive analytical stability conditions as is done in [1]. Regarding assumption ii), if $a_{0,0} > 0$, the system matrix A_b cannot be negative definite, and the network is always temporally unstable and is therefore unusable.

III. SPATIAL DYNAMICS

Note that the distribution of the node voltage v in (2.3) at an equilibrium is given by

$$A_b v_b + u_b = 0. \tag{3.1}$$

Let us first consider the spatial impulse response. If the input current is injected at node (0, 0) while no currents are injected at the other nodes, then the resultant voltage distribution is called the spatial impulse response of the network. Recall that the spatial impulse response completely characterizes a linear spatial filter network because it is identical to the convolution kernel such that any output is obtained by convolution of the input with the kernel. Suppose that the system matrix A_b is nonsingular and the input u_b is a spatial impulse δ_b

$$\delta_b := (1,0,0,\cdots,0)^T \in \mathcal{R}^{N_1N_2}.$$

Then the spatial impulse response is obtained by

$$v_b = -A_b^{-1}\delta_b. \tag{3.2}$$

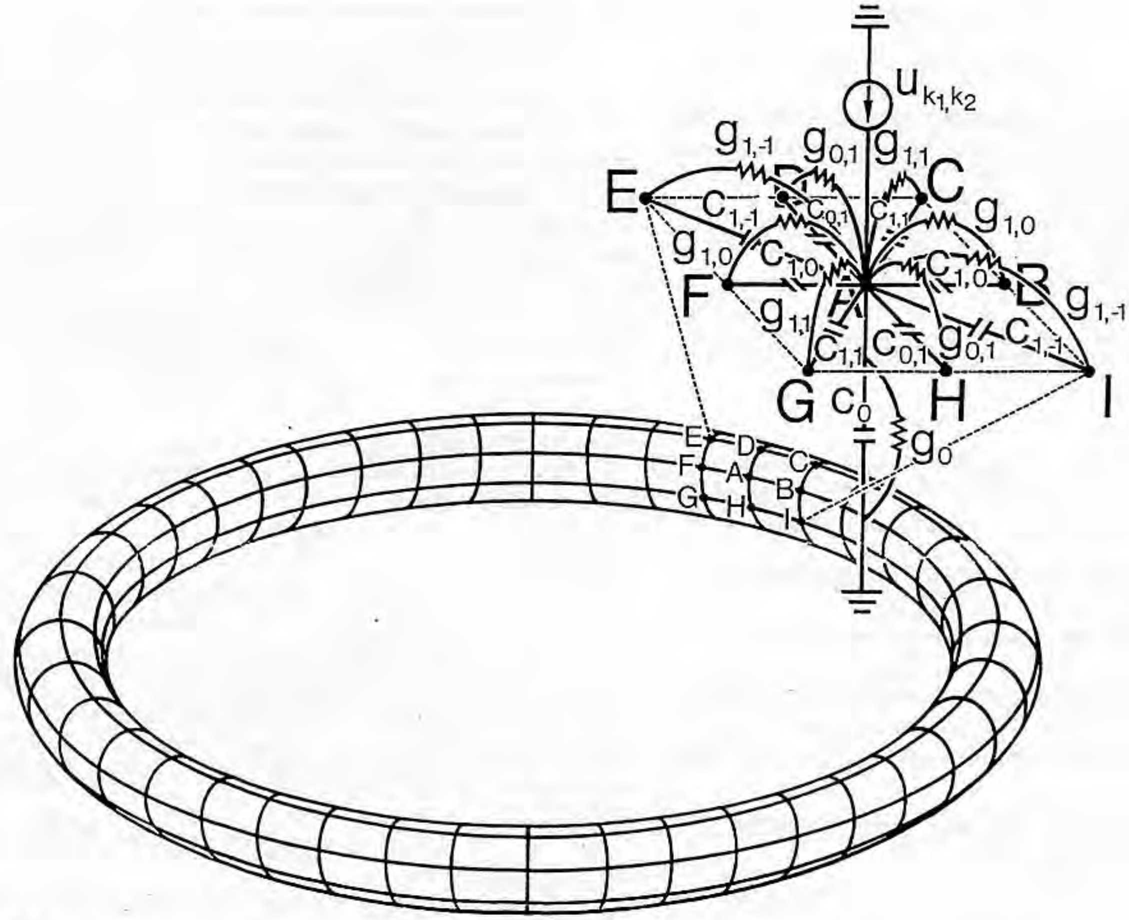


Fig. 3. A 2-D block circulant network where $m_1 = m_2 = 1$.

Proposition 1: Assume that A_b is nonsingular and let

$$c_{N_1,k_1,N_2,k_2} := (c_{N_1,k_1}^{N_2,0}, c_{N_1,k_1}^{N_2,k_2}, c_{N_1,k_1}^{N_2,2k_2}, \cdots c_{N_1,k_1}^{N_2,(N_2-1)k_2})^T \tag{3.3}$$

where

$$\begin{aligned} c_{N_1,k_1}^{N_2,k_2} := \Bigg(&\cos\left(2\pi\frac{k_2}{N_2}\right), \cos\left(2\pi\left(\frac{k_1}{N_1}+\frac{k_2}{N_2}\right)\right),\\ &\cos\left(2\pi\left(\frac{2k_1}{N_1}+\frac{k_2}{N_2}\right)\right), \cdots,\\ &\cos\left(2\pi\left(\frac{(N_1-1)k_1}{N_1}+\frac{k_2}{N_2}\right)\right)\Bigg) \end{aligned} \tag{3.4}$$

$k_1 = 0, 1, 2, \cdots, N_1 - 1$ and $k_2 = 0, 1, 2, \cdots, N_2 - 1$. Then the spatial impulse response (3.2) has the following explicit representation

$$\begin{aligned} v_b = -A_b^{-1}\delta_b = \frac{-1}{N_1N_2} \times \\ \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} \frac{1}{\displaystyle\sum_{p=-m_1}^{m_1}\sum_{q=-m_2}^{m_2} a_{p,q}\cos\left(2\pi\left(\frac{k_1p}{N_1}+\frac{k_2q}{N_2}\right)\right)} \\ \cdot c_{N_1,k_1,N_2,k_2}. \end{aligned} \tag{3.5}$$

Proof: To prove Proposition 1, note that

$$\frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\cos\left(2\pi\left(\frac{k_1n_1}{N_1}+\frac{k_2n_2}{N_2}\right)\right) = \begin{cases} 1 & (n_1 = n_2 = 0) \\ 0 & \text{otherwise.}\end{cases}$$

Thus the input impulse is represented by

$$\delta_b = \frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} c_{N_1,k_1,N_2,k_2}$$

and hence it follows from Lemma 1 (see Appendix) that

$$\begin{aligned} v_b &= \frac{-1}{N_1N_2}A_b^{-1}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} c_{N_1,k_1,N_2,k_2} \\ &= \frac{-1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\frac{1}{\lambda_{N_1,k_1,N_2,k_2}} c_{N_1,k_1,N_2,k_2}. \end{aligned} \tag{3.6}$$

□

We would like to point out that the spatial impulse response of a block circulant network is closely related to the discrete Fourier transform (DFT) and the inverse discrete Fourier transform (IDFT) of circuit parameters. Recall that the spatial impulse response of a block circulant network is given by

$$v_b = -A_b^{-1}\delta_b$$

provided that A_b is a nonsingular circulant matrix. Then $-A_b^{-1}\delta_b$ can be obtained by the 2-D DFT of circuit parameters using the following steps.

Step 1) Compute the eigenvalues of A_b, $\lambda_{N_1,0,N_2,0}$, $\lambda_{N_1,0,N_2,1}$, $\cdots$ $\lambda_{N_1,N_1-1,N_2,N_2-1}$ with the 2-D IDFT of parameters

$$\{\overbrace{a_0, a_1, \cdots, a_{m_1}, 0, \cdots, 0, a_{-m_1}, \cdots, a_{-1}}^{N_1}\}$$

where

$$a_k = (\overbrace{a_{k,0}, a_{k,1}, \cdots, a_{k,m_2}, 0, \cdots, 0, a_{k,-m_2}, \cdots, a_{k,-1}}^{N_2})^T$$
$$k = 0, \pm 1, \pm 2, \cdots, \pm m_1.$$

Step 2) Obtain the reciprocal of the eigenvalues, i.e., $\{\lambda_{N_1,0,N_2,0}^{-1}, \lambda_{N_1,0,N_2,1}^{-1}, \cdots, \lambda_{N_1,N_1-1,N_2,N_2-1}^{-1}\}$.

Step 3) Compute the spatial impulse response by the 2-D DFT of $\{\lambda_{N_1,0,N_2,0}^{-1}, \lambda_{N_1,0,N_2,1}^{-1}, \cdots, \lambda_{N_1,N_1-1,N_2,N_2-1}^{-1}\}$.

Note that the 2-D DFT and IDFT are defined as follows [11]: let $f(n_1,n_2) = f(n_1+N_1, n_2+N_2)$, where n_1 and n_2 are integers and let $F(k_1,k_2)$ be the DFT of $f(n_1,n_2)$. Then

$$F(k_1,k_2) := \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} f(n_1,n_2) W_1^{n_1k_1} W_2^{n_2k_2}$$

where $W_1 := e^{-j(2\pi/N_1)}$, $W_2 := e^{-j(2\pi/N_2)}$, $F(k_1,k_2) = F(k_1+N_1, k_2+N_2)$, and k_1, k_2 are integers. Conversely, $f(n_1,n_2)$ is the IDFT of $F(k_1,k_2)$, and

$$f(n_1,n_2) = \frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} F(k_1,k_2) W_1^{-n_1k_1} W_2^{-n_2k_2}.$$

We also remark that in general the 2-D explicit impulse response is very difficult to obtain [12].

There is an interesting way of looking at (3.5). Consider the input u_b given by

$$u_b = \alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}, \quad \text{where } \alpha, \beta \text{ are constants.} \tag{3.7}$$

Then it follows from Lemma 1 that

$$\begin{aligned} v_b &= -A_b^{-1}(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}) \\ &= \frac{-1}{\lambda_{N_1,k_1,N_2,k_2}}(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}) \end{aligned} \tag{3.8}$$

where c_{N_1,k_1,N_2,k_2}, s_{N_1,k_1,N_2,k_2} and $\lambda_{N_1,k_1,N_2,k_2}$ are given by (7.2), (7.3), and (7.1), respectively. If we regard (3.7) as a spatially periodic input function with an angle frequency $(2\pi k_1/N_1, 2\pi k_2/N_2)$, then (3.8) is the spatial frequency response of the network. Thus (3.8) can be interpreted in the following manner: for a spatially periodic input with a spatial angle frequency $(2\pi k_1/N_1, 2\pi k_2/N_2)$.

1) The gain of the spatial frequency response is $|1/\lambda_{N_1,k_1,N_2,k_2}|$, while
2) The phase of the spatial frequency response is
$$\begin{cases} 0 & \text{when } \lambda_{N_1,k_1,N_2,k_2} < 0 \\ \pi & \text{when } \lambda_{N_1,k_1,N_2,k_2} > 0. \end{cases}$$

Next, let us discuss the spatial stability of block circulant networks. Note first that Proposition 1 says that the spatial impulse response is well defined if A_b is nonsingular for a fixed network size $N_1 \times N_2$. Formula (3.5) naturally suggests that the invertibility of A_b is not enough for the spatial stability for all N_1, N_2. A problem arises when $\lambda_{N_1,k_1,N_2,k_2}$ gets smaller and smaller as the network size $N_1 \times N_2$ grows because then v_b blows up as $N_1 \times N_2$ grows, which is inappropriate for image processing purposes. To discuss the spatial stability issue valid for all N_1, N_2 as is done in [1], we will use the following definition:

Definition—Spatial Stability for a 2-D Network: The 2-D block circulant network described by (3.1) is spatially stable if, for any u_b bounded uniformly in N_1, N_2, there is a unique v_b which is bounded uniformly in N_1, N_2 and satisfies (3.1).

Of course, that u_b (respectively, v_b) bounded uniformly in N_1, N_2 means $||u_b||$ (respectively, $||v_b||$) is bounded uniformly in N_1, N_2. Thus this definition means that if the input has a finite energy $||u_b||^2$ bounded uniformly in N_1, N_2, the output energy $||v_b||^2$ is also finite and bounded uniformly in N_1, N_2.

Proposition 2: A 2-D block circulant network described by (3.1) is spatially stable if and only if A_b is negative definite uniformly in N_1, N_2.

Proof: $\Leftarrow$) This is clear from the definition of uniform negative definiteness.

$\Rightarrow$) Suppose that A_b is not negative definite uniformly in N_1, N_2, then there are two cases.

Case 1) There are N_1, k_1, N_2, and k_2 with $\lambda_{N_1,k_1,N_2,k_2} > 0$.

Case 2) For an arbitrary $\epsilon > 0$, there are N_1, k_1, N_2 and k_2, which satisfy $|\lambda_{N_1,k_1,N_2,k_2}| < \epsilon$ even if $\lambda_{N_1,k_1,N_2,k_2} < 0$ for all N_1, k_1, N_2 and k_2.

Case 2) apparently contradicts the requirement for uniform invertibility. To check case 1, let

$$f(x_1,x_2) := \sum_{p=-m_1}^{m_1}\sum_{q=-m_2}^{m_2} a_{p,q}\cos(2\pi(px_1+qx_2)). \tag{3.9}$$

Then the eigenvalues of A_b are given by $\lambda_{N_1,k_1,N_2,k_2} = f(k_1/N_1, k_2/N_2)$. Due to assumption ii) $(a_{0,0} < 0)$, A_b cannot be positive definite and therefore $f(x_{1a}, x_{2a}) < 0$ for some x_{1a}, x_{2a}. Case 1 means that there are x_{1b} and x_{2b} such that $f(x_{1b}, x_{2b}) > 0$. Then since $f(x_1,x_2)$ is continuous, by virtue of the mean value theorem there are infinitely many N_1, k_1, N_2 and k_2 with $|f(k_1/N_1, k_2/N_2)| < \epsilon$. This also contradicts the requirement for uniform invertibility. □

The following fact gives an explicit if-and-only-if condition for spatial stability in terms of network parameters. The proof immediately follows from Proposition 2 and (7.1) and (3.9).

Proposition 3: The 2-D block circulant network described by (3.1) is spatially stable if and only if

$$\sigma_+ := \max_{x_1,x_2\in[0,1]\times[0,1]} \sum_{p=-m_1}^{m_1}\sum_{q=-m_2}^{m_2} a_{p,q} \cdot \cos(2\pi(px_1+qx_2)) < 0. \tag{3.10}$$

We call σ_+ a stability indicator function.

Remark: The spatial stability condition above is consistent with the classical stability condition for noncausal discrete variable IIR filters where the network extends indefinitely. To see this, recall that the classical stability condition demands that all poles of their transfer functions are off the unit circle [12]. For example, let us consider a 2-D network where the rightmost and leftmost nodes are disconnected, the top and bottom nodes are also disconnected, and $N_1, N_2 \to \infty$. The spatial transfer function of the 2-D network can be defined in terms of

$$\frac{V(z_1, z_2)}{U(z_1, z_2)} = \frac{-1}{H(z_1, z_2)}$$

where $U(z_1, z_2)$ and $V(z_1, z_2)$ are 2-D Z-transform of u_{k_1,k_2} and v_{k_1,k_2}, respectively, [11], and

$$H(z_1, z_2) := \sum_{p=-m_1}^{m_1} \sum_{q=-m_2}^{m_2} a_{p,q} z_1^p z_2^q.$$

If the transfer function has a pole on the unit circle $|z_1| = |z_2| = 1$, there is a 2-D spatial frequency (θ_1, θ_2) which satisfies $H(e^{j\theta_1}, e^{j\theta_2}) = 0$, i.e., $|1/H(e^{j\theta_1}, e^{j\theta_2})| = \infty$, which gives the reader an intuitive interpretation of spatial instability. Since $H(e^{j\theta_1}, e^{j\theta_2}) = \sigma_+$, it is clear that this violates the stability condition of our new definition. Conversely, if all of the transfer function poles are off the unit circle, $H(e^{j\theta_1}, e^{j\theta_2})$ is nonzero for all θ_1, θ_2, which means $H(e^{j\theta_1}, e^{j\theta_2}) < 0$ for all θ_1, θ_2 or $H(e^{j\theta_1}, e^{j\theta_2}) > 0$ for all θ_1, θ_2. Due to our standing assumption ii), the second possibility is excluded and it yields $H(e^{j\theta_1}, e^{j\theta_2}) = \sigma_+ < 0$ for all θ_1, θ_2, which satisfies our stability condition.

Example 1: Let us consider a 2-D network where $m_1 = m_2 = 1$ and interconnection conductances are g_0, $g_1 := g_{0,1} = g_{0,-1} = g_{1,0} = g_{-1,0} = g_{1,1} = g_{-1,-1} = g_{1,-1} = g_{-1,1}$ (Fig. 4). The stability indicator function σ_+ is given by

$$\begin{aligned}
\sigma_+ &= \max_{x_1,x_2 \in [0,1]\times[0,1]} \{-(g_0 + 8g_1) + 2g_1(\cos(2\pi x_1) \\
&\quad + \cos(2\pi x_2) + \cos(2\pi(x_1 + x_2)) \\
&\quad + \cos(2\pi(x_1 - x_2)))\} \\
&= \begin{cases} -(g_0 + 8g_1) + 8g_1 & \text{when } g_1 \geq 0 \\ & \quad \text{at } x_1 = x_2 = 0 \\ -(g_0 + 8g_1) - 4g_1 & \text{when } g_1 < 0 \\ & \quad \text{at } x_1 = 0, x_2 = 0.5 \\ & \quad \text{or } x_1 = 0.5, x_2 = 0 \end{cases} \\
&= \begin{cases} -g_0 & \text{when } g_1 \geq 0 \\ & \quad \text{at } x_1 = x_2 = 0 \\ -g_0 - 12g_1 & \text{when } g_1 < 0 \\ & \quad \text{at } x_1 = 0, x_2 = 0.5 \\ & \quad \text{or } x_1 = 0.5, x_2 = 0. \end{cases}
\end{aligned} \tag{3.11}$$

Example 2: Let us consider a 2-D block circulant network with interconnection conductances $g_0, g_{1h} := g_{0,1} = g_{0,-1}, g_{1v} := g_{1,0} = g_{-1,0}, g_{2h} := g_{0,2} = g_{0,-2}$ and $g_{2v} := g_{2,0} = g_{-2,0}$

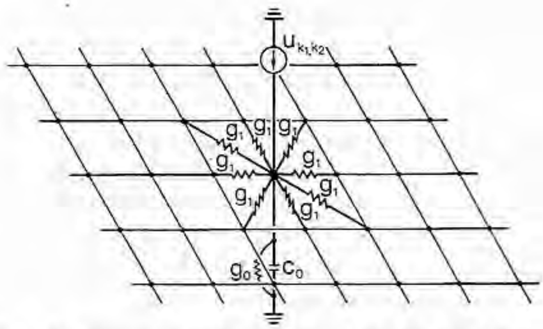


Fig. 4. A 2-D network of Examples 1 and 3. Only one unit is shown.

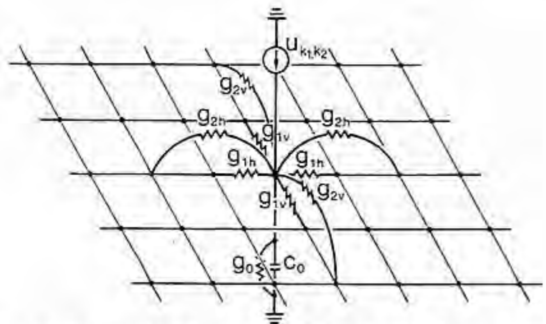


Fig. 5. A 2-D network of Example 2. Only one unit is shown.

(Fig. 5). The stability indicator function σ_+ is given by

$$\begin{aligned}
\sigma_+ &= \max_{x_1,x_2 \in [0,1]\times[0,1]} \{-(g_0 + 2g_{1h} + 2g_{1v} + 2g_{2h} + 2g_{2v}) \\
&\quad + 2g_{1h}\cos(2\pi x_1) + 2g_{1v}\cos(2\pi x_2) \\
&\quad + 2g_{2h}\cos(4\pi x_1) + 2g_{2v}\cos(4\pi x_2)\} \\
&= -(g_0 + 2g_{1h} + 2g_{1v} + 2g_{2h} + 2g_{2v}) \\
&\quad + 2\max_{x_1 \in [0,1]} \{g_{1h}\cos(2\pi x_1) + g_{2h}\cos(4\pi x_1)\} \\
&\quad + 2\max_{x_2 \in [0,1]} \{g_{1v}\cos(2\pi x_2) + g_{2v}\cos(4\pi x_2)\}
\end{aligned}$$

where

$$\begin{aligned}
&2\max_{x_1 \in [0,1]} \{g_{1h}\cos(2\pi x_1) + g_{2h}\cos(4\pi x_1)\} \\
&= \begin{cases} 2|g_{1h}| + 2g_{2h} & \text{when } g_{2h} \geq 0 \\ & \quad \text{or } \left(g_{2h} < 0 \text{ and } \left|\frac{g_{1h}}{g_{2h}}\right| \geq 4\right) \\ -2g_{2h} - \frac{g_{1h}^2}{4g_{2h}} & \text{when } g_{2h} < 0 \\ & \quad \text{and } \left|\frac{g_{1h}}{g_{2h}}\right| \leq 4 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&2\max_{x_2 \in [0,1]} \{g_{1v}\cos(2\pi x_2) + g_{2v}\cos(4\pi x_2)\} \\
&= \begin{cases} 2|g_{1v}| + 2g_{2v} & \text{when } g_{2v} \geq 0 \\ & \quad \text{or } \left(g_{2v} < 0 \text{ and } \left|\frac{g_{1v}}{g_{2v}}\right| \geq 4\right) \\ -2g_{2v} - \frac{g_{1v}^2}{4g_{2v}} & \text{when } g_{2v} < 0 \\ & \quad \text{and } \left|\frac{g_{1v}}{g_{2v}}\right| \leq 4. \end{cases}
\end{aligned}$$

Since the 2-D stability condition (3.10) is completely analytical and simple, it will be very useful to check the spatial stability of the network. One has to remember, however, that this result is for block circulant networks instead of uniform block networks. Thus, the 2-D stability test (3.10) will be of little value unless one shows that solutions for block circulant networks behave in a manner similar to those for uniform block networks. The following is an important fact which says that if a block circulant network is spatially stable, then as the network size $N_1 \times N_2$ grows, the impulse response far from the node where current is injected tends to zero so that the response behaves in a manner similar to that of a "properly" behaved solution of a uniform block network.

Proposition 4: Suppose that a block circulant network is spatially stable and current is injected to node (0, 0) while no currents are applied to the other nodes.

1) When $N_1 \to \infty$, the response v_{n_1,n_2} approaches zero for node (n_1, n_2) far from node (0, 0) with respect to the n_1-coordinate. In other words, for arbitrary $c_1 \geq 0$ and $\epsilon_1 > 0$, there is an $M > 0$ such that $|v_{n_1,n_2}| < \epsilon_1$ for all $N_1 > M$, where

$$\frac{N_1}{2} - c_1 \leq n_1 \leq \frac{N_1}{2} + c_1 \quad (n_1\text{: even}),$$
$$\frac{N_1 - 1}{2} - c_1 \leq n_1 \leq \frac{N_1 + 1}{2} + c_1 \quad (n_1\text{: odd}),$$
$$0 \leq n_2 \leq N_2 - 1.$$

2) When $N_2 \to \infty$, the response v_{n_1,n_2} approaches zero for node (n_1, n_2) far from node (0, 0) with respect to the n_2-coordinate. In other words, for arbitrary $c_2 \geq 0$ and $\epsilon_2 > 0$, there is an $M > 0$ such that $|v_{n_1,n_2}| < \epsilon_2$ for all $N_2 > M$, where

$$0 \leq n_1 \leq N_1 - 1, \quad \frac{N_2}{2} - c_2 \leq n_2 \leq \frac{N_2}{2} + c_2 \quad (n_2\text{: even}),$$
$$\frac{N_2 - 1}{2} - c_2 \leq n_2 \leq \frac{N_2 + 1}{2} + c_2 \quad (n_2\text{: odd}).$$

3) When $N_1, N_2 \to \infty$, the response v_{n_1,n_2} approaches zero for node (n_1, n_2) far from node (0, 0) with respect to the n_1-coordinate and the n_2-coordinate. In other words, for arbitrary $c_1, c_2 \geq 0$ and $\epsilon > 0$, there is an $M > 0$ such that $|v_{n_1,n_2}| < \epsilon$ for all $N_1, N_2 > M$, where

$$\frac{N_1}{2} - c_1 \leq n_1 \leq \frac{N_1}{2} + c_1 \quad (n_1\text{: even}),$$
$$\frac{N_1 - 1}{2} - c_1 \leq n_1 \leq \frac{N_1 + 1}{2} + c_1 \quad (n_1\text{: odd}),$$
$$0 \leq n_2 \leq N_2 - 1$$

or

$$0 \leq n_1 \leq N_1 - 1, \quad \frac{N_2}{2} - c_2 \leq n_2 \leq \frac{N_2}{2} + c_2 \quad (n_2\text{: even}),$$
$$\frac{N_2 - 1}{2} - c_2 \leq n_2 \leq \frac{N_2 + 1}{2} + c_2 \quad (n_2\text{: odd}).$$

Proof: Recall $f(x_1, x_2)$ defined by (3.9) and let

$$g(x_1, x_2, x_3, x_4) := \frac{\cos(2\pi(x_3 + x_4))}{f(x_1, x_2)}.$$

If the network is spatially stable, $f(x_1, x_2)$ is uniformly continuous and bounded away from zero so that $1/f(x_1, x_2)$ is uniformly continuous in $x_1, x_2 \in [0,1] \times [0,1]$. Then $g(x_1, x_2, x_3, x_4)$ is also uniformly continuous in $x_1, x_2, x_3, x_4 \in [0,1] \times [0,1] \times [0,1] \times [0,1]$. Hence, for an arbitrary $\epsilon > 0$ there is a $\delta > 0$ which satisfies

$$|g(x_1, x_2, x_3, x_4) - g(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3, x_4 + \Delta x_4)| < \epsilon \tag{3.12}$$

for all $x_1, x_2, x_3, x_4 \in [0,1] \times [0,1] \times [0,1] \times [0,1]$, $|\Delta x_1| < \delta, |\Delta x_2| < \delta, |\Delta x_3| < \delta$ and $|\Delta x_4| < \delta$. Also, if the network is stable, there is a $\xi > 0$ so that

$$|g(x_1, x_2, x_3, x_4)| < \xi \tag{3.13}$$

for all $x_1, x_2, x_3, x_4 \in [0,1] \times [0,1] \times [0,1] \times [0,1]$. Letting

$$a_1 := n_1 - \frac{N_1}{2}, \quad a_2 := n_2 - \frac{N_2}{2}$$

we obtain the impulse response at node (n_1, n_2) from (3.5)

$$\begin{aligned} v_{n_1,n_2} &= \frac{-1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \frac{1}{\lambda_{N_1,k_1,N_2,k_2}} \cdot \cos\left(2\pi\left(\frac{k_1 n_1}{N_1} + \frac{k_2 n_2}{N_2}\right)\right) \\ &= \frac{-1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \frac{1}{f(k_1/N_1, k_2/N_2)} \cdot \cos\left(2\pi\left(\frac{k_1}{N_1}\left(a_1 + \frac{N_1}{2}\right) + \frac{k_2}{N_2}\left(a_2 + \frac{N_2}{2}\right)\right)\right) \\ &= \frac{-1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \frac{(-1)^{k_1+k_2}}{f(k_1/N_1, k_2/N_2)} \cdot \cos\left(2\pi\left(\frac{k_1 a_1}{N_1} + \frac{k_2 a_2}{N_2}\right)\right) \\ &= \frac{-1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} (-1)^{k_1+k_2} g\left(\frac{k_1}{N_1}, \frac{k_2}{N_2}, \frac{k_1 a_1}{N_1}, \frac{k_2 a_2}{N_2}\right). \end{aligned} \tag{3.14}$$

We will prove only i) of Proposition 4, since proofs of ii) and iii) are similar.

When N_1 Is Even: From (3.14) we obtain

$$|v_{n_1,n_2}| \leq \frac{1}{N_1 N_2} \sum_{l_1=0}^{N_1/2-1} \sum_{k_2=0}^{N_2-1} \left| g\left(\frac{2l_1}{N_1}, \frac{k_2}{N_2}, \frac{2l_1}{N_1}a_1, \frac{k_2}{N_2}a_2\right) - g\left(\frac{2l_1+1}{N_1}, \frac{k_2}{N_2}, \frac{2l_1+1}{N_1}a_1, \frac{k_2}{N_2}a_2\right)\right|. \tag{3.15}$$

Noting that

$$\frac{N_1}{2} - c_1 \le n_1 \le \frac{N_1}{2} + c_1 \Leftrightarrow -c_1 \le a_1 \le c_1 \tag{3.16}$$

it follows from (3.12) that

$$\left| g(x_1, x_2, x_3, x_4) - g\left(x_1 + \frac{1}{N_1}, x_2, x_3 + \frac{a_1}{N_1}, x_4\right) \right| < \epsilon \tag{3.17}$$

for all N_1 with $N_1 > \max(c_1/\delta, 1/\delta)$ and all a_1, which satisfies (3.16). Then from (3.15) and (3.17) we obtain

$$|v_{n_1,n_2}| < \frac{1}{N_1 N_2} \frac{N_1}{2} N_2 \epsilon = \frac{\epsilon}{2} < \epsilon.$$

When N_1 is odd:
From (3.14) we obtain

$$\begin{aligned} |v_{n_1,n_2}| \le \frac{1}{N_1 N_2} \sum_{k_2=0}^{N_2-1} \Bigg[\Bigg\{ \sum_{l_1=0}^{(N_1-3)/2} & \cdot \left| g\left(\frac{2l_1}{N_1}, \frac{k_2}{N_2}, \frac{2l_1 a_1}{N_1}, \frac{k_2 a_2}{N_2}\right) \right. \\ & \left. - g\left(\frac{2l_1+1}{N_1}, \frac{k_2}{N_2}, \frac{2l_1+1}{N_1} a_1, \frac{k_2 a_2}{N_2}\right) \right| \Bigg\} \\ & + \left| g\left(\frac{N_1-1}{N_1}, \frac{k_2}{N_2}, \frac{N_1-1}{N_1} a_1, \frac{k_2 a_2}{N_2}\right) \right| \Bigg]. \end{aligned} \tag{3.18}$$

Noting that

$$\begin{aligned} &\frac{N_1 - 1}{2} - c_1 \le n_1 \le \frac{N_1+1}{2} + c_1 \Leftrightarrow \\ &-\left(c_1 + \frac{1}{2}\right) \le a_1 \le c_1 + \frac{1}{2} \end{aligned} \tag{3.19}$$

it follows from (3.12) and (3.13) that

$$\left| g(x_1, x_2, x_3, x_4) - g\left(x_1 + \frac{1}{N_1}, x_2, x_3 + \frac{a_1}{N_1}, x_4\right) \right| < \epsilon \tag{3.20}$$

$$|g(x_1, x_2, x_3, x_4)| < \xi < \frac{N_1 \epsilon}{2} \tag{3.21}$$

for all N_1 with $N_1 > \max(c_1/\delta, 1/\delta, 2\xi/\epsilon)$ and all a_1, which satisfies (3.19). Then from (3.18), (3.20) and (3.21), we obtain

$$|v_{n_1,n_2}| < \frac{1}{N_1 N_2} N_2 \left(\frac{N_1 - 1}{2}\epsilon + \frac{N_1}{2}\epsilon \right) < \epsilon$$

which proves 1).

One can prove 2) of Proposition 4 in a similar manner. It is clear that if 1) and 1) are valid, so is 3). □

It is known in [12] that the spatial impulse response of the 2-D network in [12] with infinite size is obtained by the inverse Z-transform [11] of the transfer function, and we will discuss the relation between this and our results.

Proposition 5: The impulse response of a stable 2-D block circulant network converges to the one obtained by the inverse Z-transform as the network size grows.

Proof: If the network is spatially stable, there are $\delta_1, \delta_2 > 0$ with $-\delta_1 < 1/H(e^{j\theta_1}, e^{j\theta_2}) < -\delta_2$ for all θ_1, θ_2 and thus the impulse response of the transfer function $-1/H(z_1, z_2)$ is given by

$$\begin{aligned} v_{k_1,k_2}^{\infty} &:= \frac{-1}{(2\pi j)^2} \oint\oint \frac{z_1^{k_1-1} z_2^{k_2-1}}{H(z_1, z_2)} dz_1\, dz_2 \\ &= \frac{-1}{(2\pi j)^2} \int_{|z_1|=1} \int_{|z_2|=1} \frac{z_1^{k_1-1} z_2^{k_2-1}}{H(z_1, z_2)} dz_1\, dz_2 \\ &= \frac{-1}{(2\pi j)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{j\theta_1(k_1-1)} e^{j\theta_2(k_2-1)}}{H(e^{j\theta_1}, e^{j\theta_2})} \\ &\quad \cdot je^{j\theta_1} je^{j\theta_2}\, d\theta_1\, d\theta_2 \\ &= \frac{-1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{j\theta_1 k_1} e^{j\theta_2 k_2}}{H(e^{j\theta_1}, e^{j\theta_2})} d\theta_1\, d\theta_2. \end{aligned} \tag{3.22}$$

Recall that from (3.5) the impulse response at node (k_1, k_2) of a 2-D block circulant network with size $N_1 \times N_2$ is given by

$$\begin{aligned} v_{k_1,k_2} &:= \frac{-1}{N_1 N_2} \sum_{p=0}^{N_1-1} \sum_{q=0}^{N_2-1} \frac{\cos\left(2\pi\left(\frac{k_1 p}{N_1} + \frac{k_2 q}{N_2}\right)\right)}{H(e^{j2\pi p/N_1}, e^{j2\pi q/N_2})} \\ &= \frac{-1}{N_1 N_2} \sum_{p=0}^{N_1-1} \sum_{q=0}^{N_2-1} \frac{e^{j2\pi(k_1 p/N_1)} e^{j2\pi(k_2 q/N_2)}}{H(e^{j2\pi p/N_1}, e^{j2\pi q/N_2})} \\ &= \frac{-1}{(2\pi)^2} \sum_{p=0}^{N_1-1} \sum_{q=0}^{N_2-1} \frac{e^{j\theta_{1,p} k_1} e^{j\theta_{2,q} k_2}}{H(e^{j\theta_{1,p}}, e^{j\theta_{2,q}})} \\ &\quad \cdot (\theta_{1,p+1} - \theta_{1,p})(\theta_{2,q+1} - \theta_{2,q}) \end{aligned} \tag{3.23}$$

where

$$\theta_{1,p} := \frac{2\pi}{N_1} p, \quad \theta_{2,q} := \frac{2\pi}{N_2} q.$$

Moreover, note that

$$\begin{aligned} 0 = \theta_{1,0} < \theta_{1,1} < \theta_{1,2} < \cdots < \theta_{1,N_1} = 2\pi, \\ 0 = \theta_{2,0} < \theta_{2,1} < \theta_{2,2} < \cdots < \theta_{2,N_2} = 2\pi \end{aligned}$$

and

$$\begin{aligned} \theta_{1,p+1} - \theta_{1,p} \to 0, \theta_{2,q+1} - \theta_{2,q} \to 0, \text{ as } N_1, N_2 \to \infty \\ \text{for } p = 0, 1, \cdots N_1 - 1, q = 0, 1, \cdots, N_2 - 1. \end{aligned}$$

If the network is stable, $1/H(e^{j\theta_1}, e^{j\theta_2})$ is continuous for $\theta_1 \times \theta_2 \in [0, 2\pi] \times [0, 2\pi]$ so that the summation in (3.23) converges to the integral in (3.22) as $N_1, N_2 \to \infty$. □

Remark: It cannot be overemphasized that Propositions 4 and 5 say that even though a 2-D block circulant network has a special boundary condition (i.e., periodic), once the stability condition (3.10) is satisfied, it behaves quite properly as the network size tends to ∞. If A_b is a uniform block matrix instead of a block circulant matrix, it is difficult to derive an analytic stability condition.

For image processing, input is naturally not always an impulse. One can show a result similar to Proposition 4 when the input is a general image, provided that its extent is uniformly bounded as N_1 and N_2 grow (see [1] for the 1-D case).

Now let us consider the 1-D cases, i.e., $N_2 = 1, m_2 = 0$ and the network size is $N_1 \times 1$. Recall our spatial stability results for uniform band networks [1], where the system matrix A_b is a uniform band matrix instead of a circulant matrix. Then $A_b v_b + u_b = 0$ can be recast as $x_{k+1} = F x_k + y_k$. It is shown in [1] that if and only if F is hyperbolic, the network is spatially stable. Since the function σ_+ defined by (3.10) is the same as the one defined in [1], Proposition 3 is consistent with the results therein. Why then, the new definition? A major obstacle in extending the 1-D spatial stability results obtained in [1] to 2-D cases lies in the fact that it is difficult to derive an F-matrix. Our new definition can handle the 2-D network as well as the 1-D network.

Note that Propositions 4 and 5 suggest that the spatial impulse response of a spatially stable circulant network is almost insensitive to changes in N_1, N_2, the network size. On the other hand, the following example demonstrates that the spatial impulse response of a spatially unstable network can be very sensitive to changes in N_1, N_2 as well as network parameter values and boundary conditions.

Example 3: Let us consider again 2-D networks where $m_1 = m_2 = 1$ and interconnection conductances are $g_0, g_1 := g_{0,1} = g_{0,-1} = g_{1,0} = g_{-1,0} = g_{1,1} = g_{-1,-1} = g_{1,-1} = g_{-1,1}$ (Fig. 4). The stability indicator function σ_+ is given by (3.11). Fig. 6(a) shows the spatial impulse response of a 2-D block circulant network with $1/g_0 = 200\,\text{k}\Omega, 1/g_1 = 20\,\text{k}\Omega$ and network size 61×61, while Fig. 6(b) shows that of a 2-D block circulant network with the same g_0 and g_1 values and network size 57×57, where $\sigma_+ < 0$ and the networks are spatially stable. Fig. 6(c) shows that of a 2-D uniform block network with the same g_0 and g_1 values and network size 61×61. Fig. 6(d) shows that of a 2-D block circulant network with $1/g_0 = 200\,\text{k}\Omega, 1/g_1 = 21\,\text{k}\Omega$ and network size 61×61 where $\sigma_+ < 0$ and the network is spatially stable. We see that these responses in Fig. 6 are similar.

On the other hand, Fig. 7(a) shows the response of a 2-D block circulant network with $1/g_0 = -200\,\text{k}\Omega, 1/g_1 = 200\,\text{k}\Omega$ and network size 61×61, while Fig. 7(b) shows that of a 2-D block circulant network with the same g_0 and g_1 values and network size 57×57, where $\sigma_+ > 0$ and the networks are spatially unstable. Fig. 7(c) shows that of a 2-D uniform block network with the same g_0 and g_1 values and network size 61×61. Fig. 7(d) shows that of a 2-D block circulant network with $1/g_0 = -200\,\text{k}\Omega, 1/g_1 = 210\,\text{k}\Omega$ and network size 61×61 where $\sigma_+ > 0$ and the network is spatially unstable. We see that these responses in Fig. 7 are very different.

This simulation suggests that the impulse responses of stable 2-D networks are relatively insensitive to changes in boundary conditions and circuit parameter values as well as network size while those of unstable 2-D networks can be very sensitive to them. We will explain why this happens. Recall (see (3.6)) that the spatial impulse response is given by

$$v_b = \frac{-1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \frac{1}{\lambda_{N_1,k_1,N_2,k_2}} c_{N_1,k_1,N_2,k_2} \tag{3.24}$$

provided that A_b is invertible. Also recall that

$$f(x_1, x_2) := \sum_{p=-m_1}^{m_1} \sum_{q=-m_2}^{m_2} a_{p,q} \cos(2\pi(px_1 + qx_2)),$$

$$\lambda_{N_1,k_1,N_2,k_2} = f(k_1/N_1, k_2/N_2). \tag{3.25}$$

Since the networks are spatially unstable, there are x_{z1}, x_{z2} which satisfy $f(x_{z1}, x_{z2}) = 0$. Then there are k_{z1}, k_{z2} such that $k_{z1}/N_1 \approx x_{z1}, k_{z2}/N_2 \approx x_{z2}, f(k_{z1}/N_1, k_{z2}/N_2) \approx 0$. By changing the network size N_1, N_2 by a small amount, $f(k_{z1}/N_1, k_{z2}/N_2)$ also changes only by a small value. Since $f(k_{z1}/N_1, k_{z2}/N_2) \approx 0$, however, the change of $1/f(k_{z1}/N_1, k_{z2}/N_2) (= 1/\lambda_{N_1,k_{z1},N_2,k_{z2}})$ can be drastic in terms of sign and amplitude. Noting (3.24) and (3.25), $(1/\lambda_{N_1,k_{z1},N_2,k_{z2}}) c_{N_1,k_{z1},N_2,k_{z2}}$ becomes a dominant component in the spatial impulse response and thus the spatial impulse response can be changed drastically by varying the network size N_1, N_2. The sensitivity of the spatial impulse response to changes in network parameters and boundary conditions can be similarly explained.

IV. Temporal Dynamics

Recall the definition of the temporal stability of 1-D networks given in [1].

Definition—Temporal Stability for a 2-D Network: The 2-D network given by (2.3) is temporally stable if A_b is negative definite for all N_1, N_2.

Proposition 6: Recall (2.4), (2.5), (2.6) and define

$$\mu_{N_1,k_1,N_2,k_2} := \sum_{p=-m_1}^{m_1} \sum_{q=-m_2}^{m_2} b_{p,q} \cdot \cos\left(2\pi\left(\frac{k_1 p}{N_1} + \frac{k_2 q}{N_2}\right)\right),$$
$$\text{where } k_1 = 0, 1, \cdots, N_1 - 1, \quad k_2 = 0, 1, \cdots, N_2 - 1.$$

Also recall

$$\lambda_{N_1,k_1,N_2,k_2} := \sum_{p=-m_1}^{m_1} \sum_{q=-m_2}^{m_2} a_{p,q} \cdot \cos\left(2\pi\left(\frac{k_1 p}{N_1} + \frac{k_2 q}{N_2}\right)\right),$$
$$\text{where } k_1 = 0, 1, \cdots, N_1 - 1, \quad k_2 = 0, 1, \cdots, N_2 - 1.$$

i) Consider the temporal dynamics (2.3) with the initial condition $v_{b0} := v_b(0)$. Then $v_b(t)$ is explicitly given by

$$\begin{aligned} v_b(t) &= e^{B_b^{-1} A_b t} v_{b0} + \int_0^t e^{B_b^{-1} A_b (t-\tau)} u_b(\tau)\, d\tau \\ &= (F_{N_1} \otimes F_{N_2})^* \operatorname{diag}(e^{-t/T_{N_1,0,N_2,0}}, e^{-t/T_{N_1,0,N_2,1}}, \\ &\qquad e^{-t/T_{N_1,0,N_2,2}}, \ldots, e^{-t/T_{N_1,N_1-1,N_2,N_2-1}}) \\ &\quad \cdot (F_{N_1} \otimes F_{N_2}) v_{b0} \\ &\quad + \int_0^t (F_{N_1} \otimes F_{N_2})^* \operatorname{diag}(e^{-(t-\tau)/T_{N_1,0,N_2,0}}, \\ &\qquad e^{-(t-\tau)/T_{N_1,0,N_2,1}}, e^{-(t-\tau)/T_{N_1,0,N_2,2}}, \ldots, \\ &\qquad e^{-(t-\tau)/T_{N_1,N_1-1,N_2,N_2-1}}) (F_{N_1} \otimes F_{N_2}) u_b(\tau)\, d\tau \end{aligned} \tag{4.1}$$

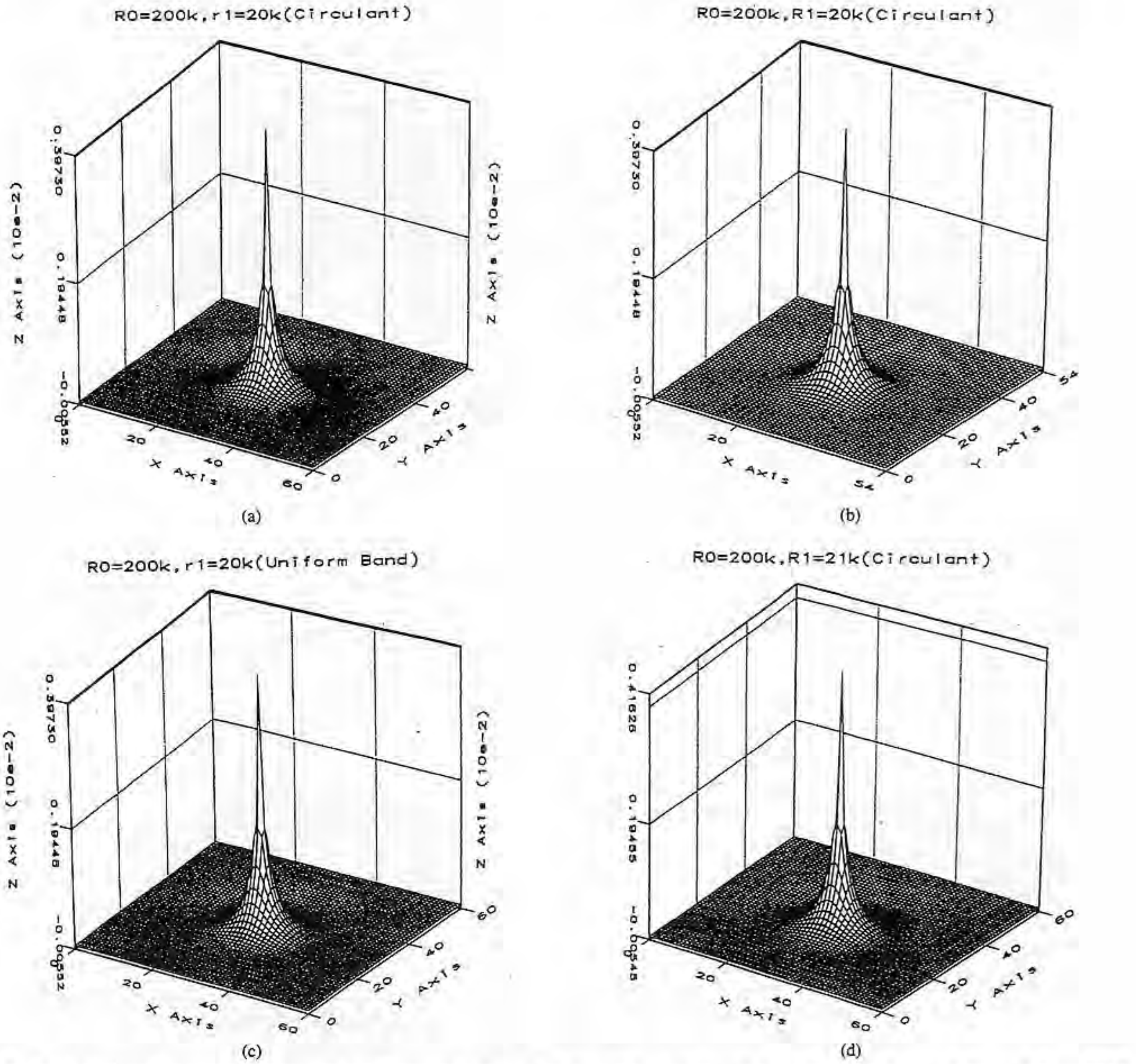


Fig. 6. Spatial impulse responses of 2-D spatially stable networks with $g_0, g_1 := g_{0,1} = g_{0,-1} = g_{1,0} = g_{-1,0} = g_{1,1} = g_{-1,-1} = g_{1,-1} = g_{-1,1}$ which corresponds to the network in Fig. 4. 1 μA current is applied to the center node. (a) Response of a block circulant network with $1/g_0 = 200\,\mathrm{k}\Omega, 1/g_1 = 20\,\mathrm{k}\Omega$ and network size 61×61. (b) Response of a block circulant network with $1/g_0 = 200\,\mathrm{k}\Omega, 1/g_1 = 20\,\mathrm{k}\Omega$ and network size 55×55. (c) Response of a uniform block network with $1/g_0 = 200\,\mathrm{k}\Omega, 1/g_1 = 20\,\mathrm{k}\Omega$ and network size 61×61. (d) Response of a block circulant network with $1/g_0 = 200\,\mathrm{k}\Omega, 1/g_1 = 21\,\mathrm{k}\Omega$ and network size 61×61.

where

$$T_{N_1,k_1,N_2,k_2} := -\frac{\mu_{N_1,k_1,N_2,k_2}}{\lambda_{N_1,k_1,N_2,k_2}}. \tag{4.2}$$

$\boldsymbol{F}_{N_1}$ and $\boldsymbol{F}_{N_2}$ are Fourier matrixes with sizes $N_1 \times N_1$ and $N_2 \times N_2$, respectively [9]. $\otimes$ denotes a Kronecker product. diag means a diagonal matrix. $(\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2})^*$ represents the conjugate transpose of $(\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2})$. And a Fourier matrix $\boldsymbol{F}_N$ with size $N \times N$ is defined by

$$\boldsymbol{F}_N := \{F(i,j)\} \in \mathcal{C}^{N\times N}, \qquad i,j = 0,1,\cdots,N_1-1$$

where

$$F(i,j) := \frac{1}{\sqrt{N}} W^{-ij}, \quad W := e^{j(2\pi/N)}$$

and $\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2}$ is given by

$$\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2} := \{\boldsymbol{F}(i,j)\} \in \mathcal{C}^{N_1N_2\times N_1N_2}, \quad i,j = 0,1,\cdots,N_1-1$$

where

$$\boldsymbol{F}(i,j) := \frac{1}{\sqrt{N_1}} W_1^{-ij} \boldsymbol{F}_{N_2} \in \mathcal{C}^{N_2\times N_2}, \quad W_1 := e^{j(2\pi/N_1)}, \quad W_2 := e^{j(2\pi/N_2)}.$$

ii) The temporal dynamics is stable if and only if

$$\sum_{p=-m_1}^{m_1} \sum_{q=-m_2}^{m_2} a_{p,q} \cos\left(2\pi(py_1 + qy_2)\right) < 0 \quad \text{for all rational } y_1, y_2; \quad 0 \le y_1; y_2 < 1. \tag{4.3}$$

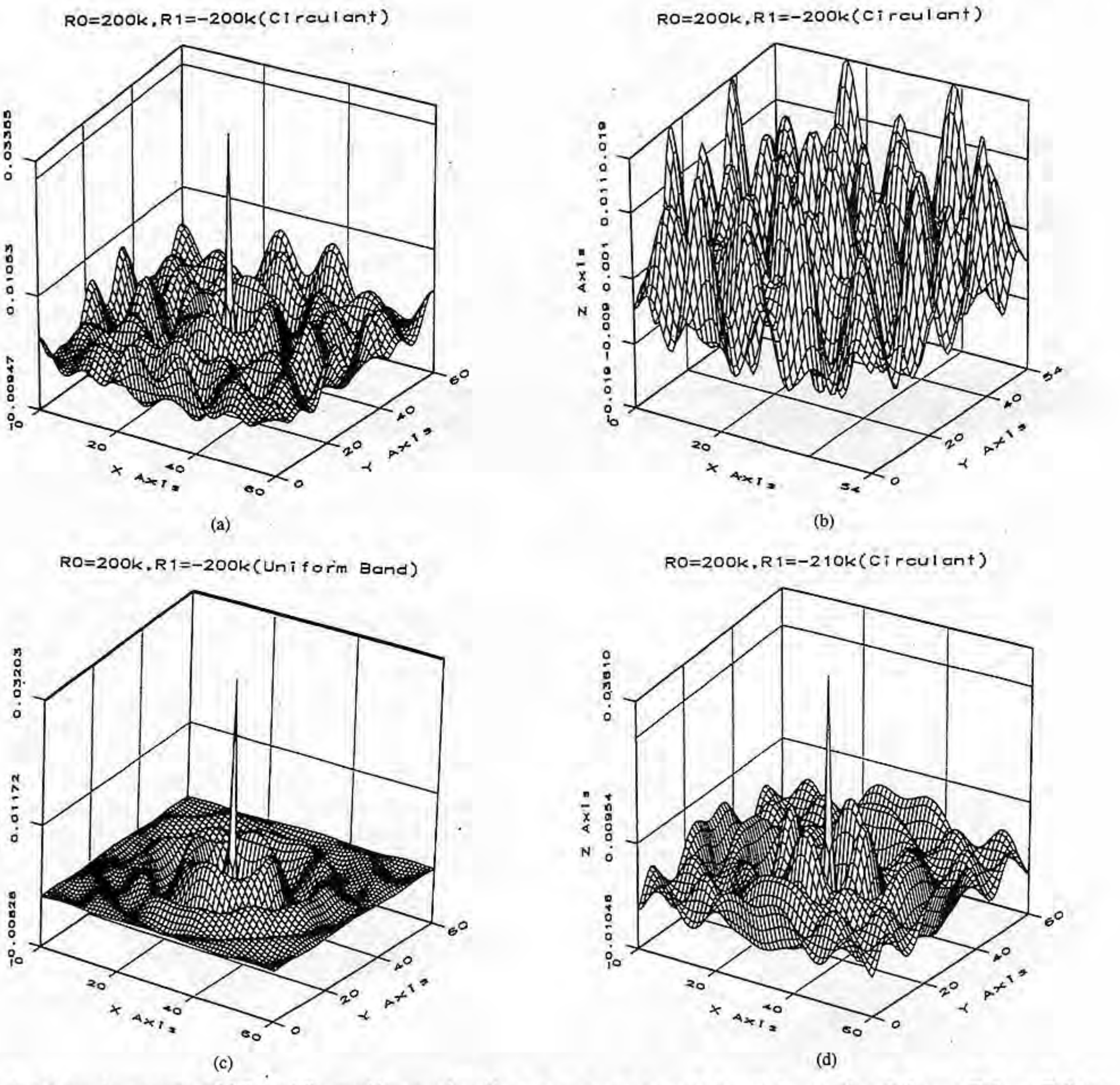


Fig. 7. Spatial impulse responses of 2-D spatially unstable networks with $g_0, g_1 := g_{0,1} = g_{0,-1} = g_{1,0} = g_{-1,0} = g_{1,1} = g_{-1,-1} = g_{1,-1} = g_{-1,1}$ which corresponds to the network in Fig. 4. (a) Response of a block circulant network with $1/g_0 = -200\,\mathrm{k}\Omega, 1/g_1 = 200\,\mathrm{k}\Omega$ and network size 61×61. (b) Response of a block circulant network with $1/g_0 = -200\,\mathrm{k}\Omega, 1/g_1 = 200\,\mathrm{k}\Omega$ and network size 55×55. (c) Response of a uniform block network with $1/g_0 = -200\,\mathrm{k}\Omega, 1/g_1 = 200\,\mathrm{k}\Omega$ and network size 61×61. (d) Response of a block circulant network with $1/g_0 = -200\,\mathrm{k}\Omega, 1/g_1 = 210\,\mathrm{k}\Omega$ and network size 61×61.

Proof: i) The solution of state equation (2.3) with the initial condition $\boldsymbol{v_{b0}} := \boldsymbol{v_b}(0)$ is given by

$$\boldsymbol{v_b}(t) = e^{\boldsymbol{B_b}^{-1}\boldsymbol{A_b}t}\boldsymbol{v_{b0}} + \int_0^t e^{\boldsymbol{B_b}^{-1}\boldsymbol{A_b}(t-\tau)}\boldsymbol{u_b}(\tau)\,d\tau. \quad (4.4)$$

From Lemma 2-i), iii), and vi) (see Appendix), it follows that

$$\begin{aligned}
\boldsymbol{B_b}^{-1}\boldsymbol{A_b}t = (\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2})^* \operatorname{diag}(&\lambda_{N_1,0,N_2,0}, \lambda_{N_1,0,N_2,1}, \cdots,\\
&\lambda_{N_1,0,N_2,N_2-1}; \lambda_{N_1,1,N_2,0}, \lambda_{N_1,1,N_2,1}, \cdots,\\
&\lambda_{N_1,1,N_2,N_2-1}; \cdots, \lambda_{N_1,N_1-1,N_2,0},\\
&\lambda_{N_1,N_1-1,N_2,1}, \cdots, \lambda_{N_1,N_1-1,N_2,N_2-1})\\
&\cdot (\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2}) \times (\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2})^*\\
&\cdot \operatorname{diag}(\mu_{N_1,0,N_2,0}, \mu_{N_1,0,N_2,1}, \cdots,\\
&\mu_{N_1,0,N_2,N_2-1}; \mu_{N_1,1,N_2,0}, \mu_{N_1,1,N_2,1}, \cdots,\\
&\mu_{N_1,1,N_2,N_2-1}; \cdots \mu_{N_1,N_1-1,N_2,0},\\
&\mu_{N_1,N_1-1,N_2,1}, \cdots, \mu_{N_1,N_1-1,N_2,N_2-1})\\
&\cdot (\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2})t\\
= (\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2})^* \operatorname{diag}(&-t/T_{N_1,0,N_1,0},\\
&-t/T_{N_1,0,N_1,1}, -t/T_{N_1,0,N_2,2}, \cdots,\\
&-t/T_{N_1,N_1-1,N_2,N_2-1})(\boldsymbol{F}_{N_1} \otimes \boldsymbol{F}_{N_2}).
\end{aligned}$$

Using Lemma 2-ii), one obtains

$$e^{B_b^{-1}A_b t}v_{b0} = (F_{N_1}\otimes F_{N_2})^* \operatorname{diag}(e^{-t/T_{N_1,0,N_2,0}}, e^{-t/T_{N_1,0,N_2,1}}, e^{-t/T_{N_1,0,N_2,2}}, \cdots, e^{-t/T_{N_1,N_1-1,N_2,N_2-1}})(F_{N_1}\otimes F_{N_2}) \cdot v_{b0}. \quad (4.5)$$

Similarly, one has

$$\int_0^t e^{B_b^{-1}A_b(t-\tau)}u_b(\tau)\,d\tau = \int_0^t (F_{N_1}\otimes F_{N_2})^* \times \operatorname{diag}(e^{-(t-\tau)/T_{N_1,0,N_2,0}}, e^{-(t-\tau)/T_{N_1,0,N_2,1}}, e^{-(t-\tau)/T_{N_1,0,N_2,2}}, \cdots, e^{-(t-\tau)/T_{N_1,N_1-1,N_2,N_2-1}})(F_{N_1}\otimes F_{N_2}) \cdot u_b(\tau)\,d\tau. \quad (4.6)$$

Equation (4.1) follows from (4.4), (4.5), and (4.6).

ii) Recall the temporal stability definition of the 2-D network. The 2-D network is temporally stable if all the eigenvalues of A_b are negative, i.e., for all N_1, k_1, N_2, and k_2

$$\lambda_{N_1,k_1,N_2,k_2} = \sum_{p=-m_1}^{m_1}\sum_{q=-m_2}^{m_2} a_{p,q} \cdot \cos\left(2\pi\left(\frac{k_1 p}{N_1}+\frac{k_2 q}{N_2}\right)\right) < 0$$

$$\text{where } k_1 = 0, 1, \cdots, N_1-1, \quad k_2 = 0, 1, \cdots, N_2-1.$$

□

Statement i) shows that T_{N_1,k_1,N_2,k_2}'s defined by (4.2) can be interpreted as "time constants" of the RC network even though "time constant" in its strictest sense is defined only for a first order network. Recall the standing assumption i) (positive definiteness of B_b) and note that μ_{N_1,k_1,N_2,k_2} is positive. Then we see that the temporal stability condition $\lambda_{N_1,k_1,N_2,k_2} < 0$ implies $T_{N_1,k_1,N_2,k_2} > 0$ for all k_1, k_2.

Remark: The assumption that "the parasitic capacitances of the resistors are the same for the similar resistors and thus B_b is a symmetric block circulant matrix" may be a rather crude approximation in actual LSI implementation. This is because parasitic capacitances of MOSFET's (metal-oxide semiconductor/field effect transistors) depend on their bias voltages and they can be mismatched during fabrication. We would like to point out, however, that this assumption led to the explicit time constant formula which can help a filter designer estimate the temporal behavior of the network.

Example 4: Consider a block circulant network where $m_1 = 2$, $m_2 = 0$, $1/g_1 := 1/g_{1,0} = 1/g_{-1,0} = 1/\cos(\sqrt{2})$ kΩ, $1/g_2 := 1/g_{2,0} = 1/g_{-2,0} = -4k\,\Omega$, $g_0 = -(2g_1 + 4g_2 + g_1^2/(4g_2))$, and the other $g_{i,j}$'s $= 0$. This network is spatially unstable but temporally stable. The stability indicator function σ_+ is given by

$$\begin{aligned}\sigma_+ :=& -(g_0+2g_1+2g_2) + 2\max_{x\in[0,1]}\cdot\{g_1\cos(2\pi x)+g_2\cos(4\pi x)\}\\ =& -(g_0+2g_1+2g_2) + 2\max_{x\in[0,1]}\cdot\left\{2g_2\left(\cos(2\pi x)+\frac{g_1}{4g_2}\right)^2-\frac{g_1^2}{8g_2}-g_2\right\}\\ =& -\left(g_0+2g_1+4g_2+\frac{g_1^2}{4g_2}\right) + 4\max_{x\in[0,1]}\cdot\left\{g_2\left(\cos(2\pi x)+\frac{g_1}{4g_2}\right)^2\right\}\\ =& 4\max_{x\in[0,1]}\left\{g_2\left(\cos(2\pi x)+\frac{g_1}{4g_2}\right)^2\right\} = 0\\ &\text{at } x = \frac{1}{2\pi}\arccos\left(-\frac{g_1}{4g_2}\right) = \frac{\sqrt{2}}{2\pi}.\end{aligned}$$

Since $\sigma_+ = 0$, it is spatially unstable. Next let us check the temporal stability. The left-hand term of (4.3) is given by

$$\sum_{p=-2}^{2} a_{p,0}\cos(2\pi p y) = 4g_2\left(\cos(2\pi y)+\frac{g_1}{4g_2}\right)^2 \le 0 \quad (4.7)$$

which is zero if and only if $y = \sqrt{2}/2\pi$. Since y is restricted to a rational number, the left-hand side of (4.7) is always negative. Thus the network is temporally stable.

We will next study several more relationships between spatial and temporal dynamics. Let $v_b(0) = 0$ and consider

$$u_b(t) = (\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})\cdot u(t),$$

$$\text{where } u(t) = \begin{cases}1 & \text{when } t \ge 0\\ 0 & \text{when } t < 0\end{cases}$$

α and β are constants, and $c_{N_1,k_1,N_2,k_2}, s_{N_1,k_1,N_2,k_2}$ are defined by (7.2) and (7.3), respectively. Then it follows from Proposition 6 that $v_b(t), t \ge 0$ is given by

$$v_b(t) = \frac{-1}{\lambda_{N_1,k_1,N_2,k_2}}(1-e^{-t/T_{N_1,k_1,N_2,k_2}})u_b(t)$$

which has a rather interesting interpretation: if there is a spatial angle frequency $(2\pi k_1/N_1, 2\pi k_2/N_2)$ for which the phase of the spatial frequency response is π, the network is temporally unstable because $\lambda_{N_1,k_1,N_2,k_2} \ge 0$; otherwise (i.e., when phase of each spatial frequency response is zero), it is temporally stable.

It is also interesting to see that the "gain" $|1/\lambda_{N_1,k_1,N_2,k_2}|$ of the spatial frequency response is proportional to the time constant $T_{N_1,k_1,N_2,k_2} = -\mu_{N_1,k_1,N_2,k_2}/\lambda_{N_1,k_1,N_2,k_2}$ of the temporal dynamics. Since μ_{N_1,k_1,N_2,k_2} and $\lambda_{N_1,k_1,N_2,k_2}$ are always real, we see that if a block circulant network is spatially stable (respectively, unstable), then the "phase" of the spatial frequency response is zero (respectively, π). We remark that a stable linear phase causal IIR (infinite impulse response) filter cannot be realized [11] but the block circulant network can be a stable zero phase noncausal IIR filter. The above fact clarifies the relation between the phase of frequency response and the stability of such noncausal IIR filters.

If the input is given by $u_b = \alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}$ [see (3.3)], the power dissipated by the network at an equilibrium is given by

$$\begin{aligned}\text{Power} &= u_b^T v_b = -(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})^T A_b^{-1} \\ &\quad \cdot (\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}) \\ &= \frac{-1}{\lambda_{N_1,k_1,N_2,k_2}} (\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})^T \\ &\quad \cdot (\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})\end{aligned}$$

where α and β are constants. Thus, when a network is spatially stable, $-\lambda_{N_1,k_1,N_2,k_2}^{-1}$ is positive and the power consumed by the network is positive, i.e., the network acts as a passive element at that spatial frequency. On the other hand, when the network is unstable, $-\lambda_{N_1,k_1,N_2,k_2}^{-1}$ is negative and the power consumed by the network is negative, i.e., the network acts as an active element at that spatial frequency.

Next consider the case that

$$\begin{aligned}u_b(t) &= g(t)(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}), \\ v_b(0) &= \gamma(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})\end{aligned}$$

where α, β, and γ are constants, and $g(t)$ is a real-valued function, such that $\int_0^t e^{-[(t-\tau)]/T_{N_1,k_1,N_2,k_2}} g(\tau)\, d\tau$ is welldefined for $t \geq 0$. Then $v_b(t)$ is given by

$$\begin{aligned}v_b(t) &= e^{B_b^{-1} A_b t} v_b(0) + \int_0^t e^{B_b^{-1} A_b (t-\tau)} B_b^{-1} u_b(\tau)\, d\tau \\ &= \Bigg(\gamma e^{-t/T_{N_1,k_1,N_2,k_2}} + \frac{1}{\mu_{N_1,k_1,N_2,k_2}} \\ &\quad \int_0^t \cdot e^{-[(t-\tau)]/T_{N_1,k_1,N_2,k_2}} g(\tau)\, d\tau\Bigg) \\ &\quad \cdot (\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}) \\ &= \xi(t)(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})\end{aligned}$$

where

$$\begin{aligned}\xi(t) := \gamma e^{-t/T_{N_1,k_1,N_2,k_2}} &+ \frac{1}{\mu_{N_1,k_1,N_2,k_2}} \\ &\int_0^t \cdot e^{-[(t-\tau)]/T_{N_1,k_1,N_2,k_2}} g(\tau)\, d\tau.\end{aligned}$$

Thus

$$\begin{aligned}B_b \frac{d}{dt} \xi(t)&(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}) \\ &= A_b \xi(t)(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}) \\ &\quad + g(t)(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}),\end{aligned}$$

$$\mu_{N_1,k_1,N_2,k_2} \frac{d}{dt} \xi(t) = \lambda_{N_1,k_1,N_2,k_2} \xi(t) + g(t). \tag{4.8}$$

Since (4.8) is the KCL equation in Fig. 8, a 2-D block circulant network, for example, in Fig. 4 is equivalent to the network shown in Fig. 8, when input $u_b(t)$ and initial value $v_b(0)$ are spatially sinusoidal. Since all the nodes are disconnected from each other in Fig. 8, one sees that $v_b = -\lambda_{N_1,k_1,N_2,k_2}^{-1} u_b$ at the equilibrium point and the time constant is $-\mu_{N_1,k_1,N_2,k_2}/\lambda_{N_1,k_1,N_2,k_2}$. Moreover, we see that the circuit theoretical interpretations of $\lambda_{N_1,k_1,N_2,k_2}$ and μ_{N_1,k_1,N_2,k_2} are that $-\lambda_{N_1,k_1,N_2,k_2}$ behaves as an effective conductance and that μ_{N_1,k_1,N_2,k_2} behaves as an effective capacitance of the network for spatially sinusoidal $u_b(t)$ and $v_b(0)$.

Example 5: The power of both the networks in Fig. 3 and Fig. 8 is given by

$$\begin{aligned}\text{Power} &= u_b^T v_b \\ &= \left(\mu_{N_1,k_1,N_2,k_2} \frac{d}{dt}\xi(t) - \lambda_{N_1,k_1,N_2,k_2}\xi(t)\right) \\ &\quad \cdot (\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})^T \xi(t) \\ &\quad \cdot (\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2}) \\ &= \mu_{N_1,k_1,N_2,k_2} \frac{d}{dt} v_b^T v_b - \lambda_{N_1,k_1,N_2,k_2} v_b^T v_b.\end{aligned}$$

The first term of the last expression indicates the power stored in the capacitor in the form of $\frac{1}{2}\mu_{N_1,k_1,N_2,k_2} v_b^T v_b$ electrostatic potential energy, while the second term indicates the power dissipated by the ohmic conductance of the network.

V. Concluding Remarks

1) In this paper we have discussed the spatio-temporal stability and dynamics of 1-D and 2-D circulant networks. We conjecture that our results can be extended for m-dimensional ($m \geq 3$) cases which may have significance in higher dimensional image-processing problems.
2) We conjecture that, in general, 1-D and 2-D spatially stable networks are relatively insensitive to changes with respect to circuit parameters $\{a_{0,0}, a_{0,1}, \cdots, a_{m1,m2}\}$, network size and boundary conditions in spatial dynamics, whereas 1-D and 2-D spatially unstable networks are very sensitive to these. We showed in [1] that the spatial impulse responses of 1-D stable networks are almost insensitive to changes with respect to network size and boundary conditions. Propositions 4 and 5 suggest that those of 2-D stable networks are robust to network size. On the other hand, Section III shows that the responses of 1-D and 2-D unstable circulant networks are very sensitive to changes with respect to network size. Also, simulation results suggest that the responses of unstable networks may vary drastically to changes with respect to circuit parameters and boundary conditions (see Example 3).
3) Hexagonal sampling is known to be the most efficient sampling method in 2-D systems [12] and it is extensively used in image-processing neuro chips [2], [7]. Our results are directly applicable to hexagonal networks as well as square networks. For example, let us consider the image-smoothing neuro chip [2] which motivated

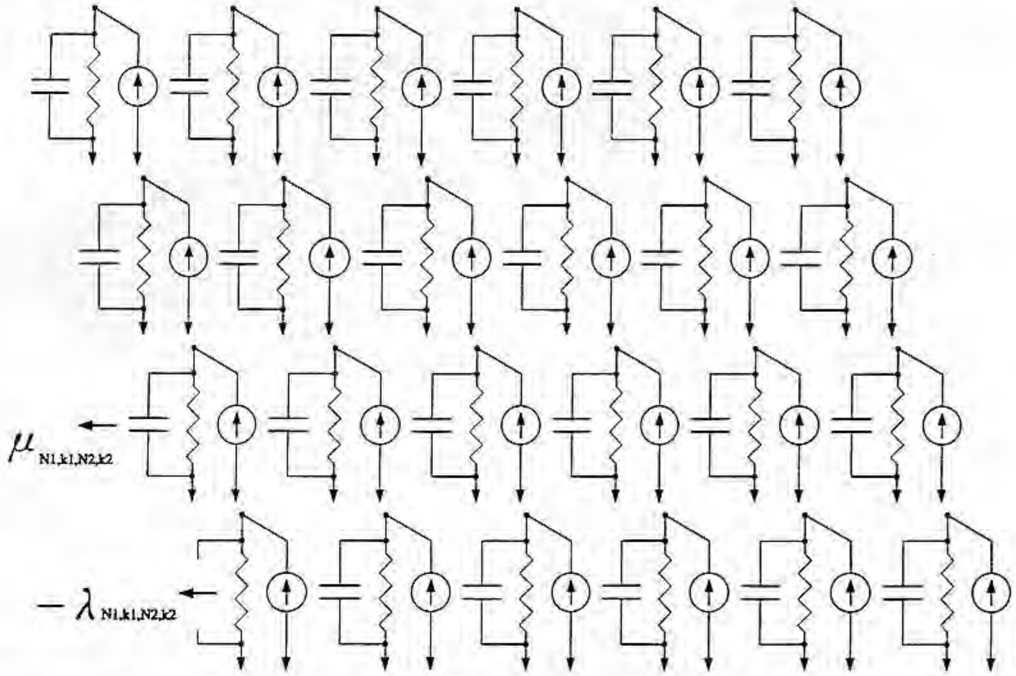


Fig. 8. An equivalent circuit for spatially sinusoidal input $u_b(t) = g(t)(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})$ and initial value $v_b(0) = \gamma\,(\alpha c_{N_1,k_1,N_2,k_2} + \beta s_{N_1,k_1,N_2,k_2})$. Each node has a conductance $-\lambda_{N_1,k_1,N_2,k_2}$ to ground and a capacitance μ_{N_1,k_1,N_2,k_2} to ground.

the present study. The structure of the network is shown in Fig. 1. There are two labeling conventions for a hexagonal grid: the standard one shown in Fig. 9(a) and the alternate one shown in Fig. 9(b). With the standard labeling convention, we obtain $g_0 > 0, g_1 := g_{1,0} = g_{0,1} = g_{1,-1} > 0, g_2 := g_{2,0} = g_{0,2} = g_{2,-2} < 0$ and $g_1 = 4|g_2|$. Thus the stability indicator function σ_+ is given by

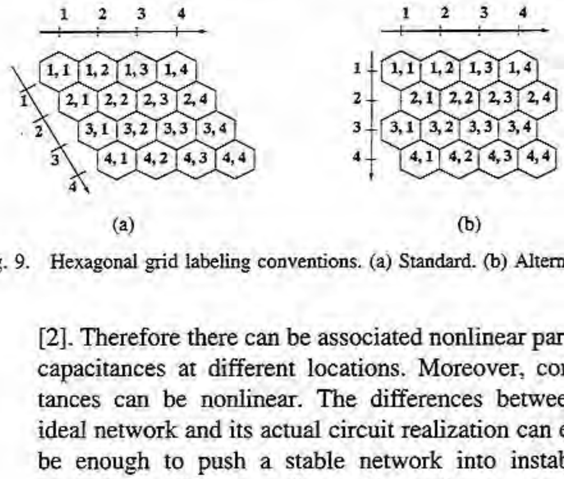
$$\sigma_+ = \max_{x_1,x_2 \in [0,1]\times[0,1]} \{-g_0 - 6g_1 - 6g_2 + 2g_1(\cos(2\pi x_1) + \cos(2\pi x_2) + 2g_1 \cos(2\pi(x_1+x_2))) + 2g_2(\cos(4\pi x_1) + \cos(4\pi x_2) + 2\cos(4\pi(x_1+x_2)))\}.$$

Noting that $g_2 = -g_1/4$, we have

$$\begin{aligned}\sigma_+ &= \max_{x_1,x_2 \in [0,1]\times[0,1]} \{-g_0 - 6g_1 + \tfrac{6}{4}g_1 + 2g_1(\cos(2\pi x_1) + \cos(2\pi x_2) + \cos(2\pi(x_1+x_2))) - \tfrac{2}{4}g_1(\cos(4\pi x_1) + \cos(4\pi x_2) + \cos(4\pi(x_1+x_2)))\}\\ &= \max_{x_1,x_2 \in [0,1]\times[0,1]} \{-g_0 - g_1\{(1-\cos(2\pi x_1))^2 + (1-\cos(2\pi x_2))^2 + (1-\cos(2\pi(x_1+x_2)))^2\}\}\\ &= -g_0 < 0 \quad \text{at } x_1 = x_2 = 0.\end{aligned}$$

Since the stability indicator function σ_+ is negative, the spatial and temporal stabilities are guaranteed.

4) This paper assumes that network conductance and capacitance are linear and a capacitor is located parallel to a conductance. In actual implementation, however, MOS transistors would be used for negative resistors [2]. Therefore there can be associated nonlinear parasitic capacitances at different locations. Moreover, conductances can be nonlinear. The differences between an ideal network and its actual circuit realization can easily be enough to push a stable network into instability, particularly if it is a fairly high-order system. The stability analysis of a network implemented with actual circuitry is important and left for future work.



(a) (b)

Fig. 9. Hexagonal grid labeling conventions. (a) Standard. (b) Alternate.

APPENDIX

This appendix gives some properties of circulant and block circulant matrixes [9].

Lemma 1—Eigenvalues and Eigenvectors of Symmetric Block Circulant Matrixes with Circulant Blocks: i) Given a network size $N_1 \times N_2$, eigenvalues $\lambda_{N_1,k_1,N_2,k_2}$ for a symmetric block circulant matrix $\boldsymbol{A_b}$ with circulant blocks are given explicitly by

$$\lambda_{N_1,k_1,N_2,k_2} = \sum_{p=-m_1}^{m_1} \sum_{q=-m_2}^{m_2} a_{p,q} \cos\left(2\pi\left(\frac{k_1 p}{N_1} + \frac{k_2 q}{N_2}\right)\right) \tag{7.1}$$

where $k_1 = 0, 1, 2, \cdots, N_1 - 1$ and $k_2 = 0, 1, 2, \cdots, N_2 - 1$.

ii) The associated eigenvectors are given by

$$c_{N_1,k_1,N_2,k_2} = (c_{N_1,k_1}^{N_2,0}, c_{N_1,k_1}^{N_2,k_2}, c_{N_1,k_1}^{N_2,2k_2}, \cdots c_{N_1,k_1}^{N_2,(N_2-1)k_2})^T \tag{7.2}$$

and

$$s_{N_1,k_1,N_2,k_2} = (s_{N_1,k_1}^{N_2,0}, s_{N_1,k_1}^{N_2,k_2}, s_{N_1,k_1}^{N_2,2k_2}, \cdots s_{N_1,k_1}^{N_2,(N_2-1)k_2})^T \tag{7.3}$$

where $c_{N_1,k_1}^{N_2,k_2}$ is given by (3.4) and $s_{N_1,k_1}^{N_2,k_2}$ is given by

$$\begin{aligned} s_{N_1,k_1}^{N_2,k_2} := \Bigg(&\sin\left(2\pi\frac{k_2}{N_2}\right), \sin\left(2\pi\left(\frac{k_1}{N_1}+\frac{k_2}{N_2}\right)\right),\\ &\sin\left(2\pi\left(\frac{2k_1}{N_1}+\frac{k_2}{N_2}\right)\right), \cdots,\\ &\sin\left(2\pi\left(\frac{(N_1-1)k_1}{N_1}+\frac{k_2}{N_2}\right)\right)\Bigg). \end{aligned} \tag{7.4}$$

Remark: In general, an eigenvalue has a unique associated eigenvector. In this case, however, since the matrixes are symmetric, each eigenvalue has a multiplicity of two (except for the case $k_1 = k_2 = 0$), i.e., $\lambda_{N_1,k_1,N_2,k_2} = \lambda_{N_1,N_1-k_1,N_2,N_2-k_2}$, and then each eigenvalue has two associated eigenvectors.

iii) When A_b is nonsingular, A_b^{-1} is also symmetric block circulant and its eigenvalues are given by

$$\lambda_{N_1,k_1,N_2,k_2}^{-1} = \frac{1}{\displaystyle\sum_{p=-m_1}^{m_1}\sum_{q=-m_2}^{m_2} a_{p,q}\cos\left(2\pi\left(\frac{k_1 p}{N_1}+\frac{k_2 q}{N_2}\right)\right)}$$

and the associated eigenvectors are also given by (7.2) and (7.3).

Lemma 2—Diagonalization, Exponential and Inverse of block circulant matrix: i) A block circulant matrix A_b with circulant blocks can be diagonalized by Fourier matrixes F_{N_1}, F_{N_2} and their conjugate transpose $F_{N_1}^*, F_{N_2}^*$

$$A_b = (F_{N_1} \otimes F_{N_2})^* \Lambda_b (F_{N_1} \otimes F_{N_2})$$

where

$$\begin{aligned} \Lambda_b := \mathrm{diag}\,(&\lambda_{N_1,0,N_2,0}, \lambda_{N_1,0,N_2,1}, \cdots,\\ &\lambda_{N_1,0,N_2,N_2-1}; \lambda_{N_1,1,N_2,0}, \lambda_{N_1,1,N_2,1}, \cdots,\\ &\lambda_{N_1,1,N_2,N_2-1}; \cdots \lambda_{N_1,N_1-1,N_2,0},\\ &\lambda_{N_1,N_1-1,N_2,1}, \cdots, \lambda_{N_1,N_1-1,N_2,N_2-1}) \end{aligned}$$

and $\lambda_{N_1,k_1,N_2,k_2}$ is an eigenvalue of A_b.

ii) If A_b is a block circulant matrix with circulant blocks, and if

$$A_b = (F_{N_1} \otimes F_{N_2})^* \Lambda_b (F_{N_1} \otimes F_{N_2})$$

then the exponential of A_b, e^{A_b}, is also a block circulant with circulant blocks and is given by

$$e^{A_b} = (F_{N_1} \otimes F_{N_2})^* e^{\Lambda_b} (F_{N_1} \otimes F_{N_2})$$

where

$$\begin{aligned} e^{\Lambda_b} = \mathrm{diag}\,(&e^{\lambda_{N_1,0,N_2,0}}, e^{\lambda_{N_1,0,N_2,1}}, \cdots,\\ &e^{\lambda_{N_1,0,N_2,N_2-1}}; e^{\lambda_{N_1,1,N_2,0}}, e^{\lambda_{N_1,1,N_2,1}}, \cdots,\\ &e^{\lambda_{N_1,1,N_2,N_2-1}}; \cdots, e^{\lambda_{N_1,N_1-1,N_2,0}},\\ &e^{\lambda_{N_1,N_1-1,N_2,1}}, \cdots, e^{\lambda_{N_1,N_1-1,N_2,N_2-1}}). \end{aligned}$$

iii) If A_b is a nonsingular block circulant matrix with circulant blocks, and if

$$A_b = (F_{N_1} \otimes F_{N_2})^* \Lambda_b (F_{N_1} \otimes F_{N_2})$$

then A_b^{-1} is also a block circulant matrix with circulant blocks and is given by

$$A_b^{-1} = (F_{N_1} \otimes F_{N_2})^* \Lambda_b^{-1} (F_{N_1} \otimes F_{N_2})$$

where

$$\begin{aligned} \Lambda_b^{-1} = \mathrm{diag}\,(&\lambda_{N_1,0,N_2,0}^{-1}, \lambda_{N_1,0,N_2,1}^{-1}, \cdots,\\ &\lambda_{N_1,0,N_2,N_2-1}^{-1}; \lambda_{N_1,1,N_2,0}^{-1}, \lambda_{N_1,1,N_2,1}^{-1}, \cdots,\\ &\lambda_{N_1,1,N_2,N_2-1}^{-1}; \cdots; \lambda_{N_1,N_1-1,N_2,0}^{-1},\\ &\lambda_{N_1,N_1-1,N_2,1}^{-1}, \cdots, \lambda_{N_1,N_1-1,N_2,N_2-1}^{-1}). \end{aligned}$$

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