Explicit Transfer Function of RC Polyphase Filter for Wireless Transceiver Analog Front-End

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Abstract – This paper derives explicit frequency transfer functions for first-, second- and third-order RC polyphase filters (which are important components in analog front-end of wireless transceivers for I, Q signal generation and image rejection) using a concept of complex signal and circulant matrix properties. The results allow us to exploit their characteristics systematically.

Keywords: Polyphase Filter, RF Circuit, Image Rejection, Wireless Transceiver, Complex Signal

I. Introduction

RC polyphase filters are important components in analog front-end of wireless transceivers; they are used for In-Phase and Quadrature (I and Q) signal generation and for image rejection [1, 2, 3, 4, 5]. However, to the best of our knowledge, up to now their design has been based on simulation [3]. In this paper we have derived their frequency transfer functions analytically, to allow their characteristics to be exploited systematically. We believe that the explicit derivation of the frequency transfer functions of second and third-order RC polyphase filters is new; those of higher-order filters could be obtained with the same approach.

II. First-Order RC Polyphase Filter

Let us consider the first-order RC polyphase filter in Fig.1 (a) and define the following:

$$\begin{aligned} I_{in}(t) &:= I_{in+}(t) - I_{in-}(t), \\ Q_{in}(t) &:= Q_{in+}(t) - Q_{in-}(t), \\ I_{out}(t) &:= I_{out+}(t) - I_{out-}(t), \\ Q_{out}(t) &:= Q_{out+}(t) - Q_{out-}(t). \end{aligned}$$

Now let us define complex signals $V_{in}(t)$ and $V_{out}(t)$ as follows [7]:

$$\begin{aligned} V_{in}(t) &:= I_{in}(t) + jQ_{in}(t), \\ V_{out}(t) &:= I_{out}(t) + jQ_{out}(t). \end{aligned}$$

Letting $V_{in}(j\omega)$, $V_{out}(j\omega)$, $I_{out}(j\omega)$, $Q_{out}(j\omega)$, $I_{in}(j\omega)$ and $Q_{in}(j\omega)$ be the Fourier transform of $V_{in}(t)$, $V_{out}(t)$, $I_{out}(t)$, $Q_{out}(t)$, $I_{in}(t)$ and $Q_{in}(t)$ respectively and so on, we have $\mathbf{v_{out}} = M_1 \mathbf{v_{in}}$. Here

$$\mathbf{v_{in}} := (I_{in+}(j\omega), Q_{in+}(j\omega), I_{in-}(j\omega), Q_{in-}(j\omega))^T,$$

$$\mathbf{v_{out}} := (I_{out+}(j\omega), Q_{out+}(j\omega), I_{out-}(j\omega), Q_{out-}(j\omega))^T,$$

$$M_1 := circl(F_1(j\omega), 0, 0, H_1(j\omega)), \tag{1}$$

($circl()$ denotes a circulant matrix [6])

$$F_1(j\omega) := \frac{1}{1 + j\omega R_1 C_1}, \quad H_1(j\omega) := \frac{-j\omega R_1 C_1}{1 + j\omega R_1 C_1}.$$

Then we obtain

$$\begin{bmatrix} I_{out}(j\omega) \\ Q_{out}(j\omega) \end{bmatrix} = \begin{bmatrix} F_1(j\omega) & -H_1(j\omega) \\ H_1(j\omega) & F_1(j\omega) \end{bmatrix} \begin{bmatrix} I_{in}(j\omega) \\ Q_{in}(j\omega) \end{bmatrix}. \tag{2}$$

We define the frequency transfer function for complex input and output signals $V_{in}(j\omega)$ and $V_{out}(j\omega)$ as follows:

$$G_1(j\omega) := \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = F_1(j\omega) + jH_1(j\omega).$$

Then we obtain the followings:

Fact 1 (i)

$$G_1(j\omega) = \frac{1 + \omega R_1 C_1}{1 + j\omega R_1 C_1}. \tag{3}$$

This frequency transfer function $G_1(j\omega)$ characterizes a first-order RC polyphase filter for complex signals.

(ii) Gain and phase are given by

$$|G_1(j\omega)| = \frac{|1 + \omega R_1 C_1|}{\sqrt{1 + (\omega R_1 C_1)^2}} \tag{4}$$

$$\angle G_1(j\omega) = -\arctan(\omega R_1 C_1). \tag{5}$$

Figs.1 (b), (c) show $|G_1(j\omega)|$ and $\angle G_1(j\omega)$ for $R_1 = 1k\Omega, C_1 = 10pF$ respectively. Also Fig.1 (d) shows Nyquist chart of $G_1(j\omega)$.

Remark (i) It follows from eq.(4) that

$$|G_1(j\omega)| \neq |G_1(-j\omega)| \quad \text{in general, and}$$

$$|G_1(j\omega)|_{\omega=0} = 1, \quad \lim_{\omega\to\pm\infty} |G_1(j\omega)| = 1,$$

$$|G_1(j\omega)|_{\omega=-\frac{1}{R_1C_1}} = 0, \quad |G_1(j\omega)|_{\omega=\frac{1}{R_1C_1}} = \sqrt{2},$$

$$\left[\frac{\partial|G_1(j\omega)|}{\partial\omega}\right]_{\omega=\frac{1}{R_1C_1}} = 0.$$

(ii) Noting that $G_1(j\omega)$ has a zero at $\omega = -1/(R_1C_1)$, we have

$$\angle G_1(j\omega) = \begin{cases} -\angle G_1(-j\omega) & (0 \le \omega < \frac{1}{R_1C_1}) \\ \angle G_1(-j\omega) - \pi & (\frac{1}{R_1C_1} < \omega) \end{cases}$$

$$\angle G_1(j\omega)|_{\omega=0} = 0, \quad \angle G_1(j\omega)|_{\omega=\frac{1}{R_1C_1}} = -\frac{\pi}{4},$$

$$\angle G_1(j\omega)|_{\omega=\frac{-1}{R_1C_1}+\Delta h, \Delta h\to+0} = \frac{\pi}{4},$$

$$\angle G_1(j\omega)|_{\omega=\frac{-1}{R_1C_1}+\Delta h, \Delta h\to-0} = -\frac{3}{4}\pi,$$

$$\angle G_1(j\omega)|_{\omega\to\pm\infty} = -\frac{\pi}{2}.$$

(ii) When

$$\begin{bmatrix} I_{out}(j\omega) \\ Q_{out}(j\omega) \end{bmatrix} = \begin{bmatrix} K(j\omega) & L(j\omega) \\ M(j\omega) & N(j\omega) \end{bmatrix} \begin{bmatrix} I_{in}(j\omega) \\ Q_{in}(j\omega) \end{bmatrix},$$

the condition for it to have the given frequency transfer function for complex signals (namely, the condition for it to be an Hilbert filter [3]) is

$$K(j\omega) = N(j\omega), \qquad L(j\omega) = -M(j\omega)$$

and eq.(2) satisfies this, which is because the matrix M_1 defined in eq.(1) is circulant [6].

It is well-known that the frequency transfer function is a Fourier transform of the impulse response when input and output are real signals; here we consider the case that they are complex signals.

Fact 2 (i) Suppose that for $I_{in}(t) = \delta(t), Q_{in}(t) \equiv 0$,

$$I_{out}(t) = g_{ii}(t), \qquad Q_{out}(t) = g_{qi}(t)$$

and for $t < 0$, $g_{ii}(t) = g_{qi}(t) \equiv 0$. Then we have

$$G_1(j\omega) = \int_{-\infty}^{\infty} (g_{ii}(t) + jg_{qi}(t))e^{-j\omega t}dt$$

because eq.(2) holds and

$$H_1(j\omega) = \int_{-\infty}^{\infty} g_{ii}(t)e^{-j\omega t}dt,$$

$$F_1(j\omega) = \int_{-\infty}^{\infty} g_{qi}(t)e^{-j\omega t}dt.$$

(ii) Suppose that for $I_{in}(t) \equiv 0, Q_{in}(t) = \delta(t)$,

$$I_{out}(t) = g_{iq}(t) \qquad Q_{out}(t) = g_{qq}(t)$$

and for $t < 0$, $g_{iq}(t) = g_{qq}(t) \equiv 0$. Then it follows from eq.(2) that

$$g_{ii}(t) = g_{qq}(t), \qquad g_{iq}(t) = -g_{qi}(t).$$

III. Second-Order RC Polyphase Filter

Fig.2 (a) shows a second-order RC polyphase filter, and we have derived its frequency transfer function $G_2(j\omega)$ explicitly using Mathematica.

Fact 3 (i)

$$G_2(j\omega) = \frac{(1 + \omega R_1C_1)(1 + \omega R_2C_2)}{1 - \omega^2 R_1C_1R_2C_2 + j\omega(C_1R_1 + C_2R_2 + 2R_1C_2)}. \tag{6}$$

(ii) Gain and phase are respectively given as follows:

$$|G_2(j\omega)| = \frac{|1 + \omega R_1C_1||1 + \omega R_2C_2|}{\sqrt{(1 - \omega^2 R_1C_1R_2C_2)^2 + \omega^2(C_1R_1 + C_2R_2 + 2R_1C_2)^2}}, \tag{7}$$

$$\angle G_2(j\omega) = \arctan\left(\frac{\omega(C_1R_1 + C_2R_2 + 2R_1C_2)}{\omega^2 R_1C_1R_2C_2 - 1}\right). \tag{8}$$

Fig.2 (b) shows $|G_2(j\omega)|$ with respect to ω, and we see that $G_2(j\omega)$ has zeros at the angular frequencies $\omega = -1/(R_1C_1)$ and $-1/(R_2C_2)$.

(iii) When $R := R_1 = R_2$ and $C := C_1 = C_2$, gain and phase are given as follows:

$$|G_2(j\omega)| = \frac{|1 + \omega RC|^2}{\sqrt{1 + 14\omega^2R^2C^2 + \omega^4R^4C^4}}. \tag{9}$$

$$\angle G_2(j\omega) = \arctan\left(\frac{4\omega CR}{\omega^2R^2C^2 - 1}\right). \tag{10}$$

Remark (i) It follows from eq.(7) that

$$|G_2(j\omega)| \neq |G_2(-j\omega)| \quad \text{in general, and}$$

$$|G_2(j\omega)|_{\omega=0} = 1, \quad \lim_{\omega\to\pm\infty} |G_2(j\omega)| = 1.$$

(ii) It follows from eqs.(9) and (10) that when $R := R_1 = R_2$ and $C := C_1 = C_2$,

$$|G_2(j\omega)|_{\omega=-\frac{1}{RC}} = 0, \quad |G_2(j\omega)|_{\omega=\frac{1}{RC}} = 1,$$

$$\angle G_2(j\omega)|_{\omega=\frac{1}{RC}} = -\frac{\pi}{2}, \quad \left[\frac{\partial |G_2(j\omega)|}{\partial\omega}\right]_{\omega=\frac{1}{RC}} = 0.$$

(iii) It follows from eqs.(3) and (6) that

$$G_2(j\omega) \neq G_1(j\omega)^2$$

even if $R := R_1 = R_2$ and $C := C_1 = C_2$.

IV. Third-Order RC Polyphase Filter

Fig.3 (a) shows a third-order RC polyphase filter, and again we have derived the frequency transfer function $G_3(j\omega)$ explicitly using Mathematica.

Fact 4 (i)

$$G_3(j\omega) := \frac{N_3(j\omega)}{D_3(j\omega)}. \tag{11}$$

Here

$$N_3(j\omega) := (1+\omega R_1C_1)(1+\omega R_2C_2)(1+\omega R_3C_3),$$

$$D_3(j\omega) := D_{3R}(\omega) + jD_{3I}(\omega),$$

$$D_{3R}(\omega) := 1 - \omega^2[R_1C_1R_2C_2 + R_2C_2R_3C_3 + R_1C_1R_3C_3 + 2R_1C_3(R_2C_2 + R_2C_1 + R_3C_2)],$$

$$D_{3I}(\omega) := \omega[R_1C_1 + R_2C_2 + R_3C_3 + 2(R_1C_2 + R_2C_3 + R_1C_3)] - \omega^3 R_1C_1R_2C_2R_3C_3.$$

(ii) Gain and phase are respectively given as follows:

$$|G_3(j\omega)| = \frac{|N_3(j\omega)|}{\sqrt{D_{3R}(j\omega)^2 + D_{3I}(j\omega)^2}} \tag{12}$$

$$\angle G_3(j\omega) = -\arctan\left(\frac{D_{3I}(j\omega)}{D_{3R}(j\omega)}\right). \tag{13}$$

Fig.3 (b) shows $|G_3(j\omega)|$ with respect to ω, and we see that $G_3(j\omega)$ has zeros at the angular frequencies $\omega = -1/(R_1C_1)$, $-1/(R_2C_2)$ and $-1/(R_3C_3)$.

(iii) When $R := R_1 = R_2 = R_3$ and $C := C_1 = C_2 = C_3$, gain and phase are given as follows:

$$|G_3(j\omega)| = \frac{|1+\omega RC|^3}{\sqrt{1 + 63\omega^2R^2C^2 + 63\omega^4R^4C^4 + \omega^6R^6C^6}}. \tag{14}$$

$$\angle G_3(j\omega) = \arctan\left(\frac{\omega^3R^3C^3 - 9\omega RC}{1 - 9\omega^2R^2C^2}\right). \tag{15}$$

Remark (i) It follows from eq.(12) that

$$|G_3(j\omega)| \neq |G_3(-j\omega)| \quad \text{in general, and}$$

$$|G_3(j\omega)|_{\omega=0} = 1, \quad \lim_{\omega\to\pm\infty} |G_3(j\omega)| = 1.$$

(ii) It follows from eqs.(14) and (15) that when $R := R_1 = R_2 = R_3$ and $C := C_1 = C_2 = C_3$,

$$|G_3(j\omega)|_{\omega=-\frac{1}{RC}} = 0, \quad |G_3(j\omega)|_{\omega=\frac{1}{RC}} = \frac{1}{\sqrt{2}},$$

$$\angle G_3(j\omega)|_{\omega=\frac{1}{RC}} = -\frac{3}{4}\pi, \quad \left[\frac{\partial |G_3(j\omega)|}{\partial\omega}\right]_{\omega=\frac{1}{RC}} = 0.$$

(iii) It follows from eqs.(3),(6) and (11) that

$$G_3(j\omega) \neq G_1(j\omega)^3, \quad G_3(j\omega) \neq G_1(j\omega)G_2(j\omega)$$

even if $R := R_1 = R_2 = R_3$ and $C := C_1 = C_2 = C_3$.

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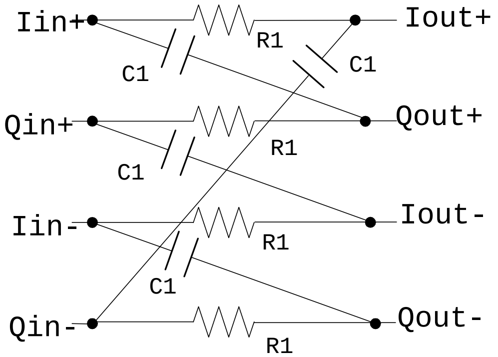

Fig.1 (a) : The first-order RC polyphase filter.

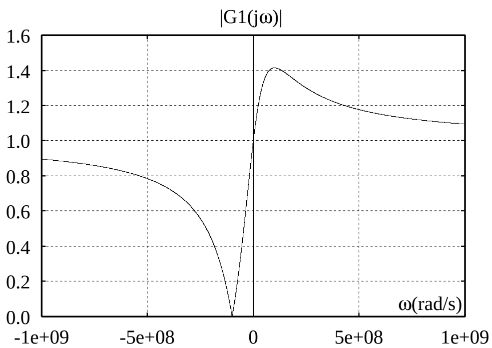

Fig.1 (b) : Gain characteristics of Fig.1 (a).

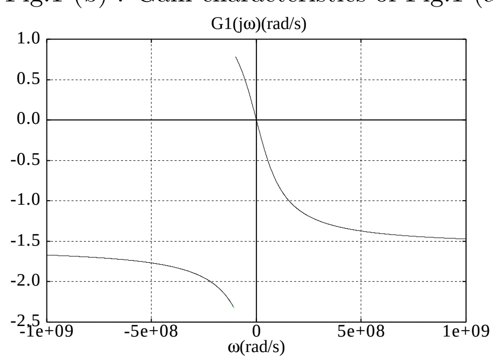

Fig.1 (c) : Phase characteristics of Fig.1 (a).

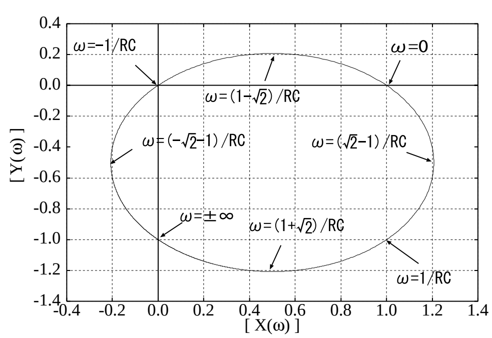

Fig.1 (d) : Nyquist chart of the first-order polyphase filter transfer function $G_1(j\omega) := X(\omega) + jY(\omega)$.

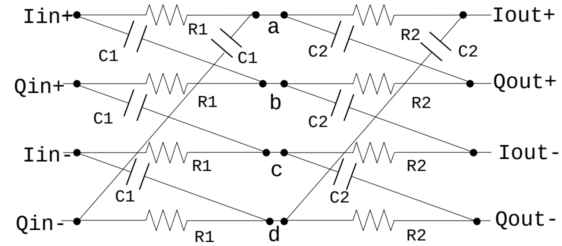

Fig.2 (a) : The second-order RC polyphase filter.

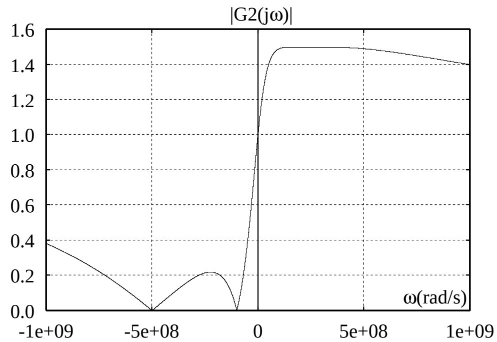

Fig.2 (b) : Gain characteristics of the second-order RC polyphase filter when $R_1 = 1k\Omega, C_1 = 10pF, R_2 = 2k\Omega$ and $C_2 = 1pF$.

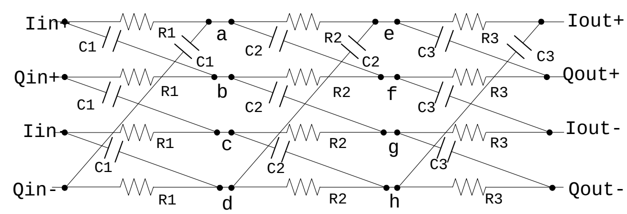

Fig.3 (a) : The third-order RC polyphase filter.

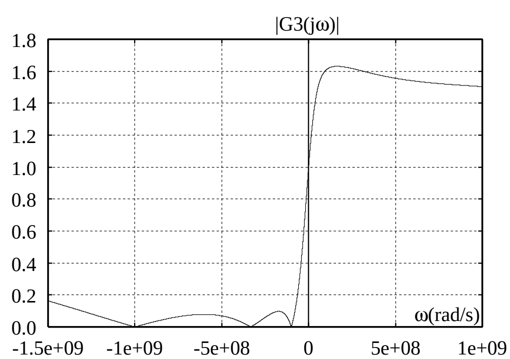

Fig.3 (b): Gain characteristics of the third-order RC polyphase filter when $R_1 = 1k\Omega, C_1 = 10pF$, $R_2 = 3k\Omega, C_2 = 1pF$, $R_3 = 5k\Omega$ and $C_3 = 0.2pF$.