Novel Linear Phase Condition for Digital Filter

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Abstract

This paper describes a linear phase digital filter with novel linear phase condition. A conventional linear phase digital filter is an FIR filter with coefficients of odd- or even –symmetry and whose group delay NTs/2 where N is the number of the FIR filter taps and Ts is the sampling period; its group delay time resolution is Ts/2. We have generalized the linear phase condition, and with our novel linear phase condition, the group delay time resolution can be arbitrary small, and the coefficients are not necessarily odd- or even-symmetric. However, ideally the number of the filter taps is infinite, and hence we have to truncate it using a window function for practical use. We present the theory of the new linear phase condition and effects of the truncation. We also compare our digital filter with the fractional delay digital filter.

Keywords: Digital Filter, Linear Phase, Window, Sinc Function, Fractional Delay Digital Filter

1. Introduction

In this paper we propose a digital filter with novel linear phase condition and show that its delay time resolution is arbitrary fine (i.e., its group delay can be set with arbitrary small time resolution). We will provide its intuitive explanation as well as rigorous proof.

Ideally, our proposed linear phase digital filter has infinite number of taps which cannot be realized. Hence we approximate it with the finite number of taps, and we will describe its truncation effects; we observe Gibbs oscillations [1-3] for phase as well as gain characteristics when we approximate it directly without applying a window function. However using proper window functions can eliminate these oscillations and their gain and phase characteristics are close to the ones with the ideal digital filter.

We also compare our linear phase digital filter with the fractional delay filter [4-7].

Our linear phase filter can be used for fine timing skew adjustment in circuits and systems since the group delay time of our linear phase digital filter is arbitrary small; we have discussed its applications in [8].

2. Conventional Linear Phase Condition

Linear phase characteristics are important for the digital filter to preserve the signal waveform in time domain. It is well-known in [1] that the FIR digital filter with odd or even symmetry coefficients has linear phase characteristics and it is unconditionally stable. The IIR digital filter with odd or even symmetry of both its denominator and nominator has also linear characteristics but it is unstable. Hence in almost all cases, the FIR digital filter with odd or even symmetry coefficients is used where the linear phase is required, and in such cases its group delay is (N/2)Ts where N is the number of the FIR filter taps and T_s is the sampling period; in other words the time resolution of the group delay is $T_s/2$, and this cannot be used for fine timing skew adjustment in ATE systems.

3. Novel Linear Phase Condition

In this section, we show the extended linear phase characteristics conditions for the digital filter which has not necessarily odd or even symmetry coefficients, and its time resolution of the group delay is arbitrary small.

First we discuss without consideration of causality, for simplicity. Let us consider the following analog filter (Fig.1):

$$v_{out} = \begin{cases} a_0 v_{in}(t) & \text{in case} -\pi/T_s < \omega < \pi/\mathrm{T_s} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Then its impulse response h(t) is given as follows:

$$h(t) = a_0 T_s \operatorname{sinc}(\pi t/T_s). \quad (2)$$

We consider the case that the input $v_{in}(t)$ is band-limited to $-\pi/T_s < \omega < \pi/\mathrm{T_s}$. We sample the above impulse response with a period Ts, and use the following transformation to obtain the digital filter which corresponds to the analog filter in (1):

$$T_s \to 1$$
$$v_{out}(nT_s) \to y(n)$$
$$v_{in}(nT_s) \to x(n).$$

(3)

Then we have the following digital filter:

$$y(n) = a_0 x(n) \tag{4}$$

This is because

$$\mathrm{sinc}(\pi n) = \begin{cases} 1 & \text{in case } n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

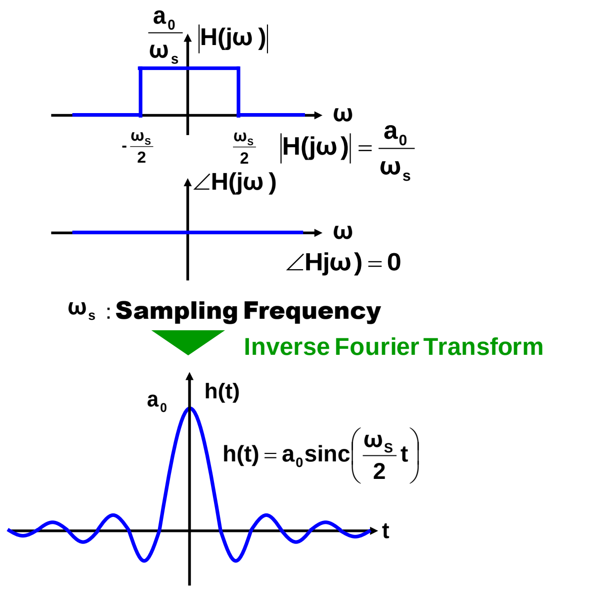

Fig. 1. An ideal analog low pass filter. Gain, phase characteristics, and impulse response.

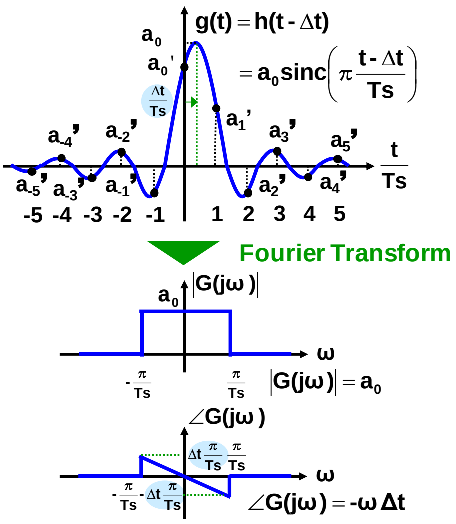

Fig. 2. Sampling timing shift can maintain the linear phase characteristics. Impulse response, and gain, phase characteristics.

This digital filter has obviously linear phase characteristics (or rigorously speaking zero phase characteristics). Now let us consider to sample $h(t)$ at $t = nT_s + \tau$ (Fig.2), where $0 < \tau < T_s$, and use (3). Then we have the following digital filter which corresponds to the analog filter in (1):

$$y(n) = \sum_{k=-\infty}^{\infty} a'_k x(n-k). \tag{6}$$

Here

$$a'_k = \mathrm{sinc}(\pi(k + \tau/T_s)). \tag{7}$$

In general a'_k is not necessarily zero and a'_k is not necessarily equal to a'_{-k} or $-a'_{-k}$.

Proposition 1 : The digital filter given by (6), (7) has the linear filter characteristics, and its group delay is τ.

Proof : The inverse Fourier transform of (6) is given by

$$Y(j\omega) = H(j\omega)X(j\omega). \tag{8}$$

It follows from (7) that in case of $\tau = 0$,

$$H(j\omega)\big|_{\tau=0} = a_0. \tag{9}$$

Then we have the following for a given τ $(0 < \tau < T_s)$:

$$H(j\omega) = a_0 e^{-j\omega\tau}.$$

(10)

Thus the digital filter given by (6), (7) has the linear filter characteristics, and its group delay is τ (Fig.2). (Q. E. D.)

Next we discuss in case of Fig.3, and consider the following analog filter:

$$v_{out} = \begin{cases} a_0 v_{in}(t) + a_1 v_{in}(t - T_s) \\ \quad \text{in case} - \pi/T_s < \omega < \pi/T \\ 0 \quad \text{otherwise.} \end{cases} \tag{11}$$

Note that its impulse response is given as follows:

$$h(t) = a_0 T_s \mathrm{sinc}(\pi t/T_s) + a_1 T_s \mathrm{sinc}(\pi(t/T_s - 1)). \tag{12}$$

We assume that the input $v_{in}(t)$ is band-limited to $-\pi/T_s < \omega < \pi/T_s$. Similarly we sample this filter with $t = nT_s$, and we have the following digital filter using (3):

$$y(n) = a_0 x(n) + a_1 x(n-1).$$

(13)

Next we sample (12) with $t = nT_s + \tau$, and we have the following digital filter:

$$y(n) = \sum_{k=-\infty}^{\infty} a'_k x(n-k).$$

(14)

Here

$$a'_k = a_0 \mathrm{sinc}(\pi(k+\tau/T_s)) + a_1 \mathrm{sinc}(\pi((k-1)+\tau/T_s)). \quad (15)$$

Proposition 2 : The digital filter given by (14), (15) with $a_0 = a_1$ or $a_0 = -a_1$ has the linear phase characteristics and its group delay is $T_s/2+\tau$. Also the digital filter of (14) has the same gain characteristics as (13).

Proof : We consider the case of $a_0 = a_1$. The inverse Fourier transform of (14) is given by

$$Y(j\omega) = H(j\omega)X(j\omega). \quad (16)$$

It follows from (15) that in case of $\tau = 0$,

$$H(j\omega)|_{\tau=0} = 2a_0 \cos(\omega T_s) e^{-j\omega T_s/2} \quad (17)$$

Then we have the following for a given τ $(0<\tau<T_s)$:

$$H(j\omega) = 2a_0 \cos(\omega T_s) e^{-j\omega(T_s/2+\tau)}. \quad (18)$$

Thus the digital filter given by (14), (15) has the linear filter characteristics, and its group delay is $(T_s/2+\tau)$.

Also it follows from (17), (18) that

$$|H(j\omega)|_{\tau=0} = |H(j\omega)|_{0<\tau<T_s} = |2a_0 \cos(\omega T_s)|. \quad (19)$$

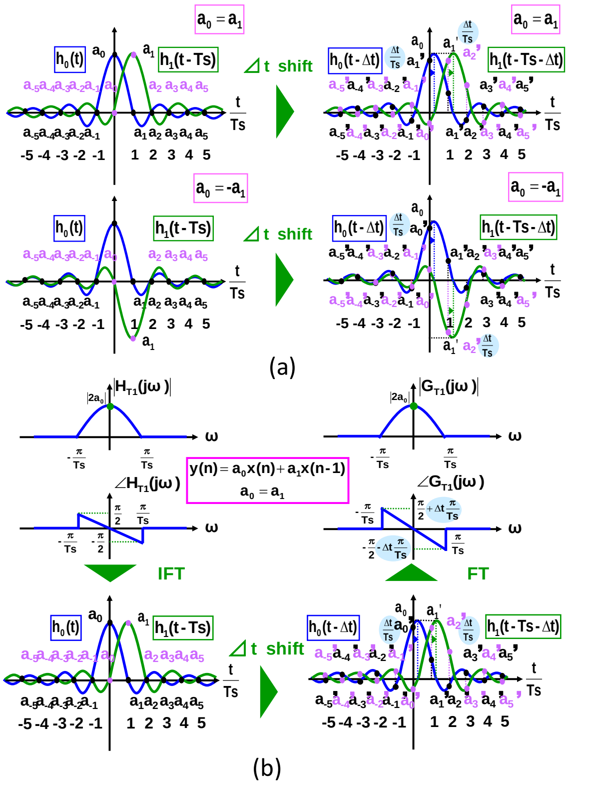

Fig. 3. 2-tap FIR filter without and with sampling timing shift. (a) Impulse response. (b) Gain and phase responses.

Then the digital filter of (14) has the same gain characteristics as (13).

Similar argument is valid in case of $a_0 = -a_1$. (Q. E. D.)

The same argument holds for the 3-tap FIR filter case (Fig.4), and also in general for an N-tap FIR filter as Propositions 1, 2.

Proposition 3 : Let us consider an N-tap FIR digital filter

with coefficients a_k of odd or even symmetry.

$$y(n) = \sum_{k=0}^{N-1} a_k x(n-k)$$

(20)

Then the following digital filter has the linear characteristics with group delay $(N/2)T_s + \tau$.

$$y(n) = \sum_{k=-\infty}^{\infty} a'_k x(n-k)$$

(21)

Here

$$a'_k = \sum_{l=0}^{N-1} a_l \mathrm{sinc}(\pi((k-l)+\tau/T_s)). \quad (22)$$

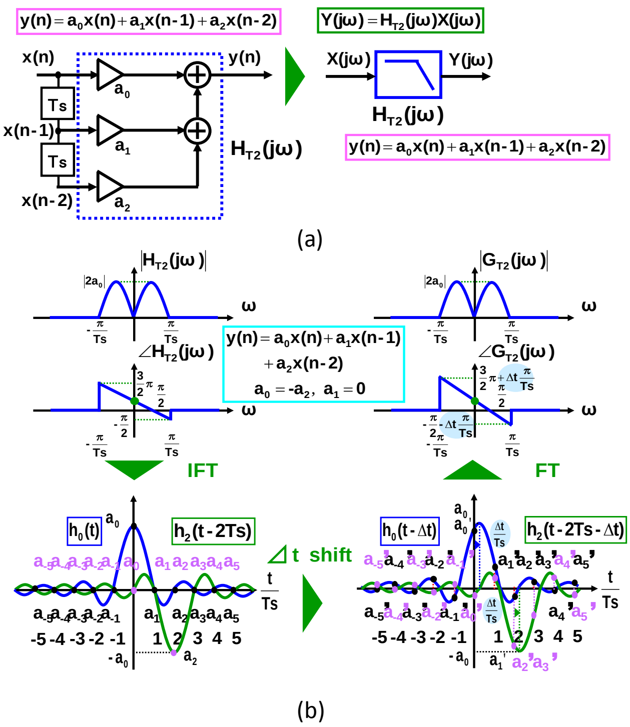

Fig. 4. 3-tap FIR filter without and with sampling timing shift. (a) Without sampling time shift. (b) Gain, phase and impulse responses with sampling timing shift.

Proposition 3 can be proved similarly in Proposition 1, 2 cases.

We have performed MATLAB simulation and checked that the above digital filter with time shift τ have linear phase characteristics and have the same gain characteristics as the one without time shift for N=1, 2, and 3.

Table 1 shows the frequency characteristics of digital filters with the conventional linear phase conditions, and Table 2 shows the ones with our proposed linear phase condition derived from the corresponding conventional linear phase digital filter.

TABLE I

FREQUENCY CHARACTERISTICS WITH CONVENTIONAL LINEAR PHASE

CONDITIONS

case	N	$h(n)$	$H(e^{j\omega})$
1	odd	even symmetry	$e^{-j\omega\frac{N-1}{2}}\sum_{k=0}^{\frac{N-1}{2}} a_k\cos(k\omega)$
2	even	even symmetry	$e^{-j\omega\frac{N-1}{2}}\sum_{k=0}^{\frac{N}{2}} a_k\cos((k-\frac{1}{2})\omega)$
3	odd	odd symmetry	$e^{-j\omega\frac{N-1}{2}}\sum_{k=0}^{\frac{N-1}{2}} a_k\sin(k\omega)$
4	even	odd symmetry	$e^{-j\omega\frac{N-1}{2}}\sum_{k=0}^{\frac{N}{2}} a_k\sin((k-\frac{1}{2})\omega)$

TABLE II

FREQUENCY CHARACTERISTICS WITH PROPOSED LINEAR PHASE

CONDITIONS

case	N	$h(n)$	$H(e^{j\omega})$
1	odd	even symmetry	$e^{-j\omega(\frac{N-1}{2}+\frac{\tau}{T_s})}\sum_{k=0}^{\frac{N-1}{2}} a_k\cos(k\omega)$
2	even	even symmetry	$e^{-j\omega(\frac{N-1}{2}+\frac{\tau}{T_s})}\sum_{k=0}^{\frac{N}{2}} a_k\cos((k-\frac{1}{2})\omega)$
3	odd	odd symmetry	$e^{-j\omega(\frac{N-1}{2}+\frac{\tau}{T_s})}\sum_{k=0}^{\frac{N-1}{2}} a_k\sin(k\omega)$
4	even	odd symmetry	$e^{-j\omega(\frac{N-1}{2}+\frac{\tau}{T_s})}\sum_{k=0}^{\frac{N}{2}} a_k\sin((k-\frac{1}{2})\omega)$

Now we will provide the proof for our proposed linear phase digital filter in general case:

Let us consider an N-tap FIR filter with conventional linear phase condition, and we have the impulse response $\tilde{h}(t)$ with continuous time and its Fourier transform $\tilde{H}(f)$:

$$\tilde{h}(t)=\sum_{n=0}^{N-1}h(nT_s)\delta(t-nT_s). \quad (23)$$

$$\tilde{H}(f)=H(f)\star\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{T_s}).$$

(24)

Here $\star$ indicates convolution, $\delta(\cdot)$ denotes a delta function, and

$$H(f)=|H(f)|e^{-j2\pi f\frac{N-1}{2}T_s} \quad (-\frac{1}{2T_s}\le f\le\frac{1}{2T_s}). \quad (25)$$

When we add a delay τ to the impulse response $h(t)$ and we have its frequency characteristics as follows:

$$H'(f)=|H(f)|e^{-j2\pi f\frac{N-1}{2}T_s}\cdot e^{-j2\pi f\tau}. \quad (26)$$

We see that the phase characteristics of $H'(f)$ is linear with respect to f. $H'(f)$ can be interpreted as the convolution between $H(f)$ and $S(f)$, where $S(f)$ is the ideal filter with a delay τ:

$$S(f)=e^{-j2\pi f\frac{N-1}{2}\tau} \quad (-\frac{1}{2T_s}\le f\le\frac{1}{2T_s}). \quad (27)$$

Thus the ideal filter $\tilde{S}(f)$ for (24) is given by

$$\tilde{S}(f)=S(f)\star\sum_{k=-\infty}^{\infty}\delta(f-\frac{k}{T_s})=\sum_{k=-\infty}^{\infty}S(f-\frac{k}{T_s}). \quad (28)$$

Next we will consider the effect of the delay τ to the impulse response. The inverse Fourier transform of $\tilde{S}(f)$ is given as follows:

$$\begin{aligned}\tilde{s}(t)&=\mathrm{sinc}(\frac{\pi(t-\tau)}{T_s})\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\\&=\sum_{n=-\infty}^{\infty}\mathrm{sinc}(\frac{\pi(t-\tau)}{T_s})\delta(t-nT_s).\end{aligned}$$

(29)

We see from (29) that $\tilde{s}(t)$ is asymmetric with respect to $t=0$, and we have the following impulse response:

$$\begin{aligned}&\tilde{h}(t)\star\tilde{s}(t)\\&=\sum_{n=0}^{N-1}h(nT_s)\delta(t-nT_s)\star\sum_{n=-\infty}^{\infty}\mathrm{sinc}(\frac{\pi(kT_s-\tau)}{T_s})\cdot\delta(t-nT_s)\\&=\sum_{n=-\infty}^{\infty}\sum_{n=0}^{N-1}h(nT_s)\mathrm{sinc}(\frac{\pi(kT_s-\tau)}{T_s})\cdot\delta(t-(n-k)T_s). \quad (30)\end{aligned}$$

Thus the impulse response of time delay τ with continuous time has finite values for $t\to\pm\infty$ due to the *sinc* function effects.

4. Realization Consideration

Here we sample the input signal with the sampling period Ts and then we consider the band-limited case to $-\pi/T_s<\omega<\pi/T_s$, in order to avoid the aliasing effects. In such case $h(t)$ does not converge to zero as t becomes plus/minus infinity. So the digital filter with our novel linear phase condition has to have the infinite number of taps and this cannot be realized. (Note that in case of $\tau=0$, $h(nT_s)$ can be zero as n becomes large which corresponds to the conventional linear phase FIR digital filter case.) So we consider to truncate the terms for large number of $|k|$ in (22) applying a window function and we approximate the digital filter of (21), (22) with the finite number of taps.

4.1 Approximation with Finite Number of Taps

The ideal digital filter with our proposed linear phase condition needs infinite number of taps. However it is cannot be realized, and hence we have to approximate it as the filter with finite number of taps. We consider here the effects of the truncation to the finite number of taps. We observe from our simulation results so-called Gibbs oscillation at the edges of pass-band of the gain characteristics and also phase characteristics (Fig.5) [1], [2]; Gibbs oscillation for phase characteristics is not observed in many cases, and we have found that this Gibbs oscillation for phase characteristics is due to the asymmetry of the impulse response $h(n)$ with respect to $n=0$.

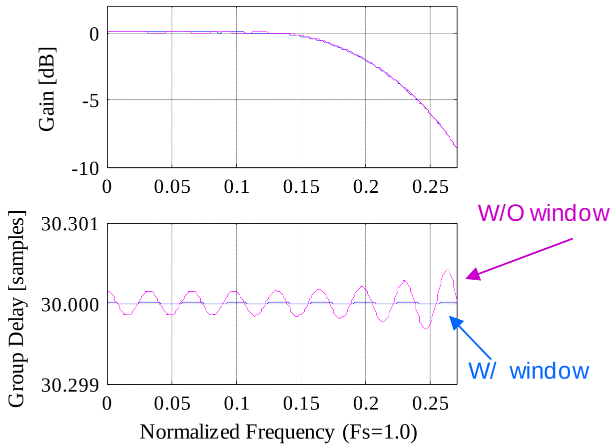

Fig. 5. Gain and phase characteristics of the proposed digital filter (with time shift τ of 0.3Ts) after truncation to finite number (N=61) of filter taps with and without applying Hann window.

4.2 Applying Window Function

Next we investigate to use window functions when we approximate the ideal filter using the one with the finite number of taps. When we use a window function, the Gibbs oscillations for gain and phase are suppressed. Fig.5 shows our simulation result with time-shift τ of 0.3Ts and applying Hann window. We have also found that this Gibbs oscillation for phase can be further suppressed if we use a window function with the time-shift τ, as shown in Fig.6 where we choose the time shift τ of 0.5Ts (which affects phase characteristics significantly) and we use a Hann window.

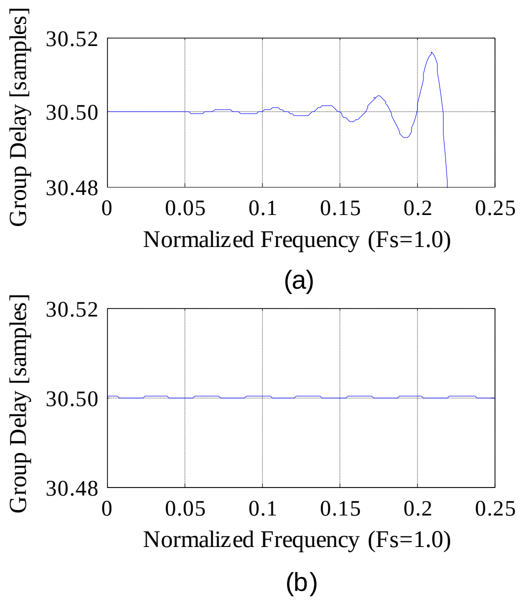

Fig. 6. Phase characteristics of the proposed digital filter (with time shift τ of $0.5T_s$) after truncation to finite number (N = 61) of filter taps. (a) With applying Hann window of no time-shift. (b) With applying Hann window time-shifted by $\tau = 0.5T_s$

5. Comparison with Fractional Delay Filter

We would like to call the reader's attention that another digital filter with fine time resolution, so-called a fractional delay digital filter has been proposed [5-7], which mainly focuses on the waveform interpolation and reconstruction. However our proposed technique can incorporate filtering characteristics (such as a cosine roll-off filter) as well as fine timing skew adjustment with the clear design method as described above; this is very useful in some electronic manufacturing equipment applications [8]. Furthermore, since our proposed filter is easy to design, we can obtain their coefficient values with small amount of calculation which is desirable for some applications where real-time timing calibration is required.

We have performed Matlab simulation and found that our proposed filter can apply for the signal upto the frequency close to the Nyquist rate (in other words, the bandwidth of our proposed filter is close to the Nyquist rate while that of the fractional delay filter is not); this is another advantage of our proposed filter (Fig.7).

6. Conclusion

We have proposed the digital filter with novel linear phase characteristics and the time resolution of its group delay is arbitrary small. We have shown its theory. We have also investigate the truncation effects to the finite number of its filter taps. We believe that our proposed digital filter will open a new research area for digital filters with linear phase and fine resolution of group delay, as well as its applications.

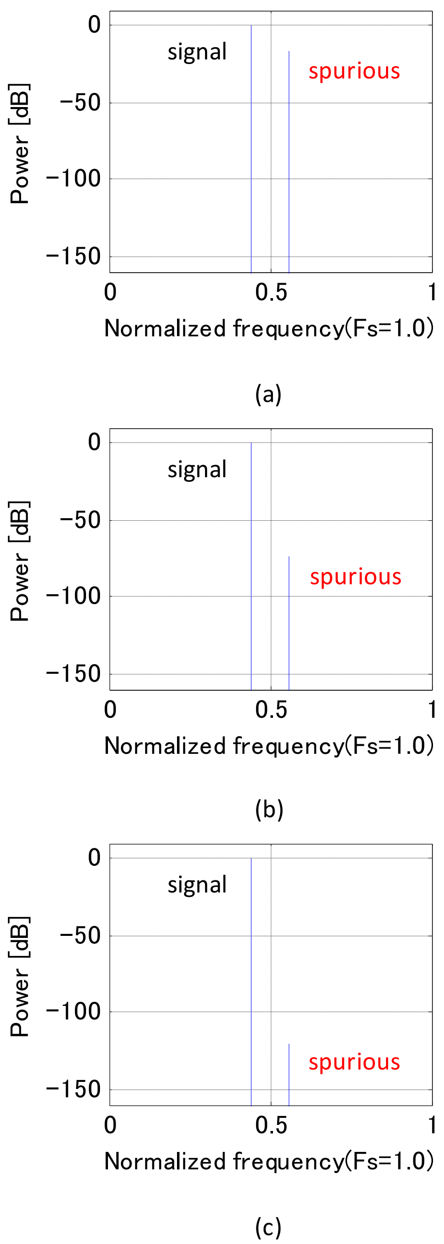

Fig.7: SSB signal power spectrum Matlab simulation results. (a) Without compensation. (b) With compensation using a 301-tap fractional delay filter and a (symmetric) blackman window. (c) With compensation using our proposed 301-tap digital filter and the same blackman window.

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