Flat Passband Gain Design Algorithm for 2nd-order RC Polyphase Filter

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Abstract – This paper describes a design algorithm of a 2^{nd}-order RC polyphase filter to obtain its flat passband gain. The condition for its solution is shown and the image rejection ratio formula is also derived. Its effectiveness is demonstrated by numerical calculation in several cases.

Keywords - RC Polyphase Filter, Flat Gain, Image Rejection, Complex Signal

1. Introduction

RC polyphase filters are important components in analog front-ends of wireless transceivers; they are used for In-Phase and Quadrature (I and Q) signal generation and for image rejection [1,2]. RF circuit designers prefer to choose the 2^{nd}-order RC polyphase filter for practical application because it is effective compared to the 1^{st}-order one and not so complicated compared to the higher order ones, In this paper we propose an explicit design algorithm for a flat-passband 2^{nd}-order RC polyphase filter; 3 equations are given for 4 variables (R_1, R_2, C_1, C_2) and hence one design freedom is left for the designer. Also the image rejection ratio is given when the proposed algorithm is used. Numerical simulations in several cases demonstrate the effectiveness of the proposed algorithm.

2. Transfer Function

Let us consider a 2^{nd}-order RC polyphase filter in Fig.1 and define the following:

$\mathrm{I_{in}(t) := I_{in+}(t) - I_{in-}(t)}$, $\mathrm{Q_{in}(t) := Q_{in+}(t) - Q_{in-}(t)}$

$\mathrm{I_{out}(t) := I_{out+}(t) - I_{out-}(t)}$, $\mathrm{Q_{out}(t) := Q_{out+}(t) - Q_{out-}(t)}$.

Define complex signals $\mathrm{V_{in}(t)}$ and $\mathrm{V_{out}(t)}$ as follows:

$$\mathrm{V_{in}(t) := I_{in}(t) + jQ_{in}(t)}$$

$$\mathrm{V_{out}(t) := I_{out}(t) + jQ_{out}(t)}.$$

Letting $\mathrm{V_{in}}(j\omega), \mathrm{V_{out}}(j\omega)$ be the Fourier transform of $\mathrm{V_{in}(t), V_{out}(t)}$, and then the frequency transfer function $G_2(j\omega)$ is obtained as follows:

$$G_2(j\omega) := \frac{\mathrm{V_{out}}(j\omega)}{\mathrm{V_{in}}(j\omega)}$$

$$= \frac{(1 + \omega R_1 C_1)(1 + \omega R_2 C_2)}{1 - \omega^2 R_1 C_1 R_2 C_2 + j\omega(R_1 C_1 + R_2 C_2 + 2R_1 C_2)}. \quad (1)$$

We define as follows:

$$\omega_1 := \frac{1}{R_1 C_1}, \ \omega_2 := \frac{1}{R_2 C_2}, \ \omega_{12} := \frac{1}{R_1 C_2}, \ \omega_{21} := \frac{1}{R_2 C_1}.$$

We see that $G_2(j\omega)$ has zeros at $-\omega_1, -\omega_2$.

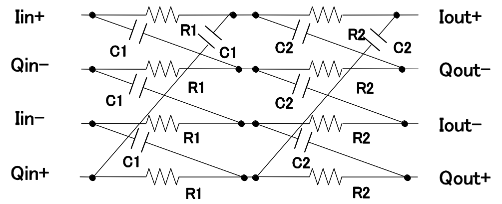
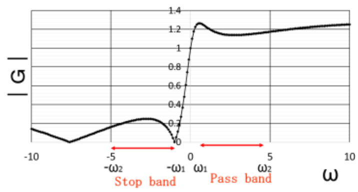


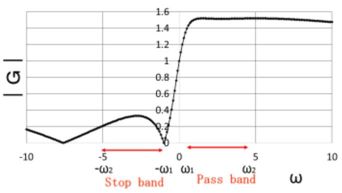
Fig.1: 2nd-order RC polyphase filter.

3. Proposed Design Algorithm

In this section, we propose a design algorithm for 2nd-order filters to make their passband gain flat. Let us consider design of a 2nd-order RC polyphase filter; we will determine the four parameter values of R_1, R_2, C_1 and C_2. For example, look at Fig.2, where the stopband is between $-\omega_2$ and $-\omega_1$ and the passband is between ω_1 and ω_2 with ω_1 = 1.0, ω_2 = 7.58. We see that the gain in the passband is not flat in Fig.2 (a) with ω_{21} = 2.0, but it is flat in Fig.2 (b) with ω_{21} = 0.44 which is obtained by our proposed algorithm.



(a) ω_{21} = 2.0. Gain in the passband is NOT flat.



(b) ω_{21} = 10.44. Gain in the passband is flat.

Fig.2: Gain characteristics of a 2nd-order RC polyphase filter with ω_1 = 1.0, ω_2 = 7.58.

Problem Formulation:

<u>Filter Specification</u>

Stopband: $-\omega_a < \omega < -\omega_b$

Passband: $\omega_b < \omega < \omega_a$

Restriction: $\frac{\omega_a}{\omega_b} < \delta$. Here $\delta = 12.63556$.

Choose values of R_1, R_2, C_1 and C_2 to make $|G_2(j\omega)|$ very flat in the passband.

<u>Proposed Design Algorithm</u>

Choose values of R_1, R_2, C_1 and C_2 as follows:

$$\frac{1}{R_1C_1} = \omega_a,\ \frac{1}{R_2C_2} = \omega_b,\ \frac{1}{R_2C_1} = \frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha} \tag{2}$$

or

$$\frac{1}{R_1C_1} = \omega_b,\ \frac{1}{R_2C_2} = \omega_a,\ \frac{1}{R_2C_1} = \frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}. \tag{3}$$

Here

$$\alpha := 6\omega_1^2 + 6\omega_2^2 + 4\omega_1\omega_2 - 8\sqrt{\omega_1\omega_2}(\omega_1 + \omega_2) \tag{4}$$

$$\beta := 6\omega_1^3 + 6\omega_2^3 + 10\omega_1\omega_2(\omega_1 + \omega_2) - 8\sqrt{\omega_1\omega_2}(\omega_1 + \omega_2)^2 \tag{5}$$

$$\gamma := \omega_1^4 + \omega_2^4 + 2\omega_1\omega_2(\omega_1^2 + \omega_2^2 + 5\omega_1\omega_2) - 4\sqrt{\omega_1}\omega_2(\omega_1^3 + \omega_1^2\omega_2 + \omega_1\omega_2^2). \tag{6}$$

The number of parameters (R_1, R_2, C_1 and C_2) is 4 while the number of their constraint equations (eq.(2) or eq.(3)) is 3. So the designer can add one more constraint arbitrarily, for example, by considering their physical implementation.

Now we will explain why the proposed algorithm can make the passband gain flat. Note that the transfer function of the 2nd-order filter is given by eq.(1), and its gain and phase are given as follows:

$$|G_2(j\omega)| = \sqrt{X_2(j\omega)^2 + Y_2(j\omega)^2} = \frac{\left|1+\frac{\omega}{\omega_1}\right|\left|1+\frac{\omega}{\omega_2}\right|}{\sqrt{\left(1-\frac{\omega^2}{\omega_1\omega_2}\right)^2 + \omega^2\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{2}{\omega_{12}}\right)^2}}. \tag{7}$$

$$\tan\angle G_2(j\omega) = X_2(j\omega)/Y_2(j\omega) = \frac{\omega\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{2}{\omega_{12}}\right)}{\omega^2/(\omega_1\omega_2) - 1}$$

Here $G_2(j\omega) := X_2(\omega) + jY_2(\omega)$.

$$X_2(\omega) := \frac{\left(1+\frac{\omega}{\omega_1}\right)\left(1+\frac{\omega}{\omega_2}\right)\left(1-\frac{\omega^2}{\omega_1\omega_2}\right)}{\left(1-\frac{\omega^2}{\omega_1\omega_2}\right)^2 + \omega^2\left(\frac{1}{\omega_1}+\frac{1}{\omega_1}+\frac{2}{\omega_{12}}\right)^2}$$

$$Y_2(\omega) := -\frac{\omega\left(1+\frac{\omega}{\omega_1}\right)\left(1+\frac{\omega}{\omega_2}\right)\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{2}{\omega_{12}}\right)}{\left(1-\frac{\omega^2}{\omega_1\omega_2}\right)^2 + \omega^2\left(\frac{1}{\omega_1}+\frac{1}{\omega_1}+\frac{2}{\omega_{12}}\right)^2}.$$

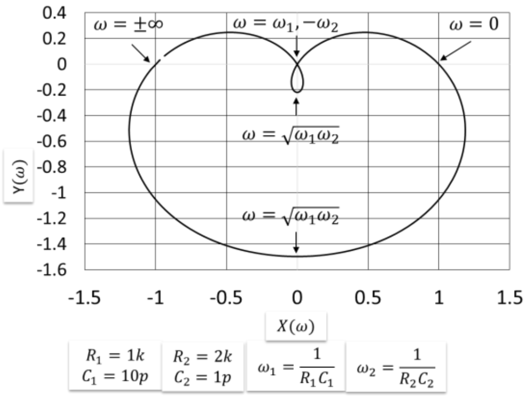
We plot the Nyquist chart of $G_2(j\omega)$ in Fig.3 (a), and we have found the followings:

(i) The Nyquist chart of $G_2(j\omega)$ is symmetric with respect to Y-axis.

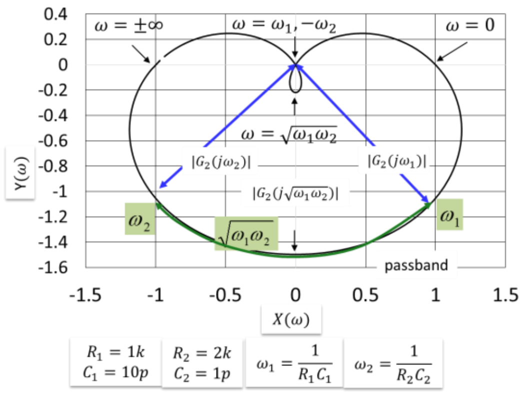
(ii) $\tan\angle G_2(-j\sqrt{\omega_1\omega_2}) = \tan\angle G_2(j\sqrt{\omega_1\omega_2}) = -\frac{\pi}{2}$

(iii) $\tan\angle G_2(j\omega_1) = -\tan\angle G_2(j\omega_2)$

Its characteristics is shown also in Table I.



(a)



(b)

Fig.3: Nyquist chart of a 2nd-order RC polyphase filter. (a) Nyquist chart of $G_2(j\omega)$. (b) Characteristics of $G_2(j\omega)$ Nyquist chart. We see that $|G_2(j\omega_1)| = |G_2(j\omega_2)|$, and $G_2(j\sqrt{\omega_1\omega_2})$ has only imaginary part and no real part. $G_2(j\omega)$ parts from ω_1 to $\sqrt{\omega_1\omega_2}$ and from $\sqrt{\omega_1\omega_2}$ to ω_2 are symmetric with respect to $G_2(j\sqrt{\omega_1\omega_2})$. (Note that $G_2(j\omega)$ from ω_1 to ω_2 represents the passband.)

Table 1: Characteristics of $|G_2(j\omega)|$.

ω	$X_2(\omega)$	$Y_2(\omega)$	$\lvert G_2(j\omega)\rvert$
$-\infty$	-1.0	0.0	1.0
$-\omega_1$	0.0	0.0	0.0
$-\omega_2$	0.0	0.0	0.0
$-\sqrt{\omega_1\omega_2}$	0.0	$-$p	p
0.0	1.0	0.0	1.0
$\sqrt{\omega_1\omega_2}$	0.0	$-$q	q
ω_1	r	s	t
ω_2	$-$r	s	t
∞	-1.0	0.0	1.0

$$\mathrm{p} := \frac{(\sqrt{\omega_1} - \sqrt{\omega_2})^2}{\omega_1 + \omega_2 + 2\omega_{21}}, \quad \mathrm{q} := \frac{(\sqrt{\omega_1} + \sqrt{\omega_2})^2}{\omega_1 + \omega_2 + 2\omega_{21}}$$

$$\mathrm{r} := \frac{(\omega_1 + \omega_2)(-\omega_1 + \omega_2)}{\omega_1^2 + \omega_1\omega_2 + \omega_2^2 + \omega_{12}(\omega_1 + \omega_2 + \omega_{12})},$$

$$\mathrm{s} := -\frac{(\omega_1 + \omega_2)(\omega_1 + \omega_2 + 2\omega_{12})}{\omega_1^2 + \omega_1\omega_2 + \omega_2^2 + \omega_{12}(\omega_1 + \omega_2 + \omega_{12})},$$

$$\mathrm{t} := \frac{\sqrt{2}(\omega_1 + \omega_2)}{\sqrt{\omega_1^2 + \omega_2^2 + 2\omega_{21}(\omega_1 + \omega_2 + \omega_{21})}}.$$

It follows from eq.(7) that

$$|G_2(j\omega_1)| = |G_2(j\omega_2)|. \qquad (8)$$

We propose the following condition to make the gain flat in the passband. For given ω_1 and ω_2, choose $\omega_{21} := 1/(R_2C_1)$ such that

$$|G_2(j\omega_1)|(= |G_2(j\omega_2)|) = |G_2(j\sqrt{\omega_1\omega_2})|. \qquad (9)$$

It follows from eq.(8) that the gain is the same at ω_1, ω_2 which are terminals of the passband (Fig.3 (b)). Also we see from Fig.3 (a), (b) that $G(j\omega)$ is symmetric with respect to Y-axis for $\omega_1 < \omega < \sqrt{\omega_1\omega_2}$ and $\sqrt{\omega_1\omega_2} < \omega < \omega_2$. Then it is expected from Fig.3 (b) that $|G(j\omega)|$would be almost constant for $\omega_1 \leq \omega \leq \omega_2$.

Now let us solve eq. (9);

$$|G_2(j\sqrt{\omega_1\omega_2})| = \frac{(\sqrt{\omega_1} + \sqrt{\omega_2})^2}{\omega_1 + \omega_2 + 2\omega_{21}}.$$

Then we have a quadratic equation for ω_{21}

$$\alpha\omega_{21}^2 + \beta\omega_{21} + \gamma = 0.$$

Here α, β, γ are defined in eqs.(4), (5), (6).

Since

$$\alpha \geq 2(\omega_1 - \omega_2)^2 > 0,$$
$$\beta \geq 2(\omega_1 - \omega_2)^2(\omega_1 + \omega_2) > 0,$$

then, if $\gamma < 0$, a positive real solution is given by

$$(\omega_{21})_+ = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

We investigate the sign of γ which depends on the values of ω_1, ω_2 using numerical calculation.

Define $f(c) := \gamma/\omega_2^4$, and we have

$$\mathrm{f(c)} = \mathrm{c}^4 + 1 + 2\mathrm{c}(\mathrm{c}^2 + 5\mathrm{c} + 1) - 4\sqrt{\mathrm{c}}(\mathrm{c}^3 + \mathrm{c}^2 + \mathrm{c} + 1)$$

Here $c := \frac{\omega_1}{\omega_2}$. Fig. 4 shows the plot of $\mathrm{f}(c)$, and we found from Fig.4 that the condition of $f(c) < 0$ (i.e, $\gamma < 0$) is as follows:

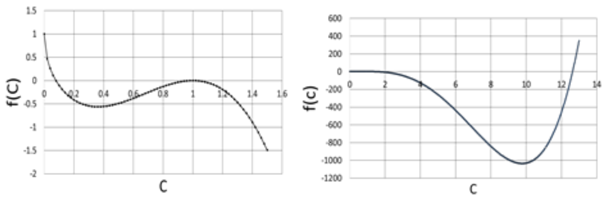
$$1/\delta < \omega_1/\omega_2 < \delta. \qquad (10)$$

Note that $1/\delta = 0.079142\ldots, \delta = 12.63556\ldots$.

Under this condition, we have a positive real solution of eq. (10) as follows:

$$\omega_{21} = \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}.$$

The proposed algorithm provides 3 restrictions to 4 parameters of R_1, C_1, R_2, C_2, and then one freedom is left for the designer.



(a) (b)

Fig.4 : f(c) characteristics. (a) $0<c<1.5$. (b) $0<c<13$.

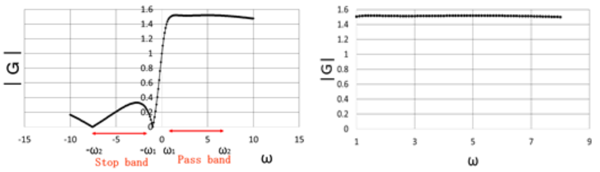
$f(c) < 0$ when $0.079142.. < c < 12.63556\ldots$.

4. Numerical Simulation for Proposed Algorithm

We have performed numerical simulations with several examples to demonstrate the effectiveness of the proposed algorithm. The values of ω_1, ω_2 are the same as in Fig.2, and we have obtained ω_{21} = 10.44 using the proposed algorithm and its gain characteristics is shown in Fig.5; we see that the gain in Fig.5 is flat compared to the one in Fig.2 (a) in the band $(1.0 < \omega < 7.58)$.

Fig.6 shows the gain characteristics for several values of ω_{21} with the same ω_1, ω_2 in Fig.2. We see that the gain is the most flat for $\omega_{21} = 10.44$ which is obtained by the proposed method.

Fig.7 shows the gain characteristics for 2 cases of ω_1, ω_2 using the values of ω_{21} obtained by the proposed algorithm and we see that in both cases, the gain is flat.



(a) (b)

Fig.5: (a) Gain characteristics of a 2nd-order RC polyphase filter with $\omega_1 = 1.0$, $\omega_2 = 7.58$, $\omega_{21} = 10.44$. The value 0.44 of ω_{21} is derived by our proposed algorithm. (b) Gain characteristics in the passband of a 2nd-order RC polyphase filter with $\omega_1 = 1.0$, $\omega_2 = 7.58$, $\omega_{21} = 10.44$.

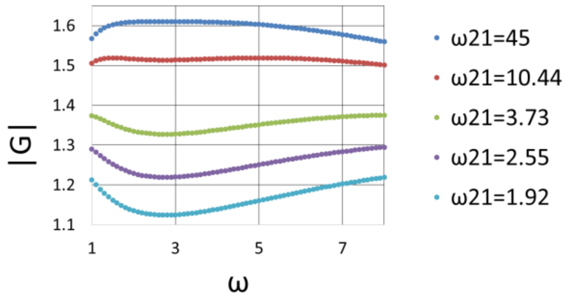
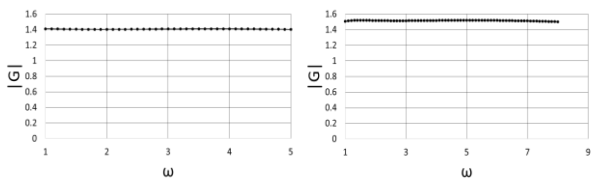


Fig.6: Gain characteristics in the passband of 2nd-order RC polyphase filters with $\omega_1 = 1.0$, $\omega_2 = 7.58$. From the top to the bottom lines, $\omega_{21} = 45, 10.44, 3.73, 2.55, 1.92$.



(a) (b)

Fig.7 : Gain characteristics in the passband ($\omega_1 < \omega < \omega_2$) of 2nd-order RC polyphase filters. Each value of ω_{21} is derived by our proposed algorithm. (a) In case of $\omega_1 = 1.0$, $\omega_2 = 3.58$, $\omega_{21} = 2.58$. (b) In case of $\omega_1 = 1.0$, $\omega_2 = 7.58$, $\omega_{21} = 10.44$.

5. Image Rejection Ratio

We define Image Rejection Ratio (IIR) for the 2nd-order RC polyphase filter as follows:

$$\mathrm{IRR} := 20 \log_{10} \frac{\text{Gain in passband}}{\text{Ripple in stopband}} \ [\mathrm{dB}]$$

When the proposed algorithm is used, the gain in the passband is given by

$$|G_2(j\omega)|_{\omega_b < \omega < \omega_a} \approx |G_2(j\sqrt{\omega_1 \omega_2})|.$$

Fig.3 shows that ripple in stopband is given by

$$\max_{-\omega_a < \omega < -\omega_b} |G_2(j\omega)| = |G_2(-j\sqrt{\omega_1 \omega_2})|.$$

In case the proposed algorithm is used, we have

$$\mathrm{IRR} = 20 \log_{10} \frac{|G_2(j\sqrt{\omega_1 \omega_2})|}{|G_2(-j\sqrt{\omega_1 \omega_2})|}$$

$$= 20 \log_{10} \frac{(\sqrt{\omega_1} + \sqrt{\omega_2})^2}{(\sqrt{\omega_1} - \sqrt{\omega_2})^2} \ [\mathrm{dB}].$$

The ratio of ω_1 / ω_2 must be close to one, in order to make IIR lager, and then the condition for $\gamma < 0$ is not severe.

6. Conclusion

In this paper, we have shown an explicit design algorithm for flat passband gain of 2nd-order RC polyphase filters and demonstrated its effectiveness using several numerical examples. We have shown an explicit condition for its solution to exist, and also an explicit image rejection ratio when our algorithm is employed.

As a future project, we intend to extend this method to higher order RC polyphase filters.

References

[1] H. Kobayashi, J. Kang, T. Kitahara, S. Takigami, H. Sadamura,"Explicit Transfer Function of RC Polyphase Filter for Wireless Transceiver Analog Front-End, "2002 IEEE Asia-Pacific Conference on ASICs, pp.137-140, Taipei, Taiwan (Aug. 2002).

[2] F. Behbahani, Y. Kishigami, J. Leete and A. A. Abidi,"CMOS Mixers and Polyphase Filters for Large Image Rejection," IEEE J. Solid-State Circuits, vol.36, no.6, pp.873-887 (June 2001).

[3] K. W. Martin, "Complex Signal Processing is NOT Complex", IEEE Trans. Circuits and Systems I, vol.51, no.9 pp.1823-1836 (Sept. 2004).

[4] H. San, Y. Jingu, H. Wada, H. Hagiwara, A. Hayakawa, H. Kobayashi, et. al., "A Second-Order Multi-bit Complex Bandpass ΔΣ AD Modulator With I, Q Dynamic Matching and DWA Algorithm", IEICE Trans. vol.E90-C, no.6, pp.1181-1188 (Jun. 2007).