Analysis and Design of Operational Amplifier Stability Based on Routh-Hurwitz Method

〇王建龍
Gopal Adhikari　小林春夫　築地伸和　平野繭　栗原圭汰
群馬大学大学院　理工学府電子情報部門

長浜顕仁　野田一平　吉井宏治
リコー電子デバイス(株)

Contents

Contents

Research Background (Stability Theory)

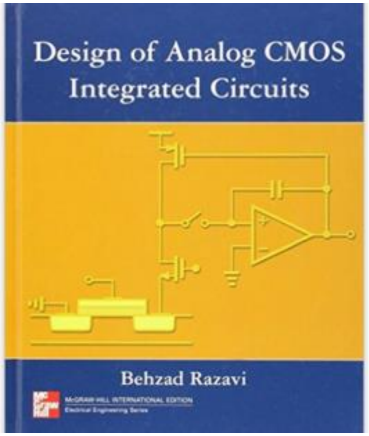
- **Electronic Circuit Design Field**
  - Bode plot (>90% frequently used)
  - Nyquist plot (源代裕治、電子回路研究会 2015年7月)

- **Control Theory Field**
  - Bode plot
  - Nyquist plot
  - Nicholas plot
  - Routh-Hurwitz stability criterion
    ⇨ Very popular in control theory field
    but rarely seen in electronic circuit books/papers
  - Lyapunov function method

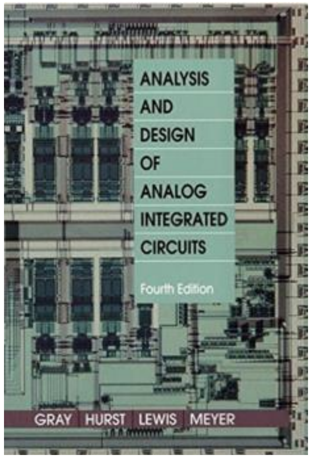
:

Electronic Circuit Text Book

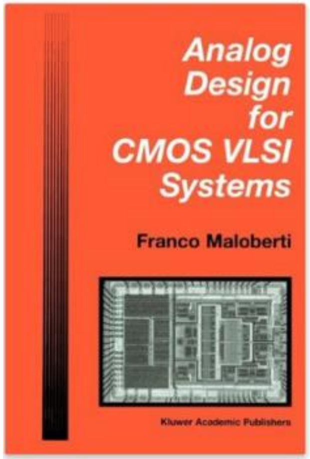
We were NOT able to find out any electronic circuit text book which describes Routh-Hurwitz method
for operational amplifier stability analysis and design !



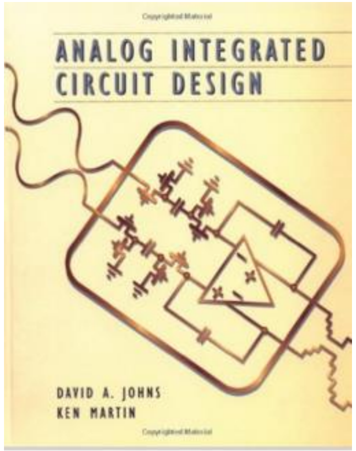
Razavi



Gray



Maloberti

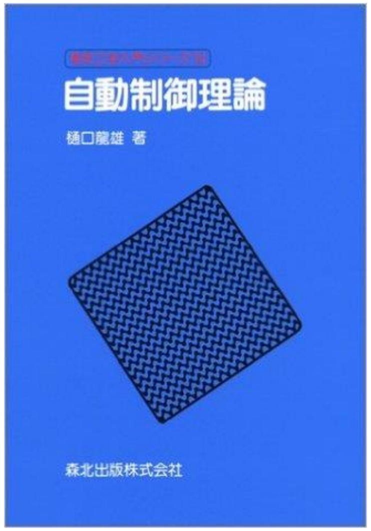


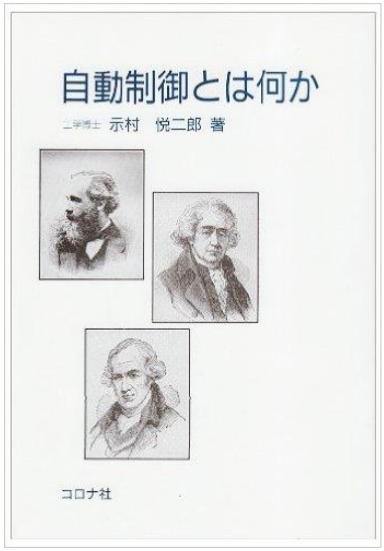
Martin

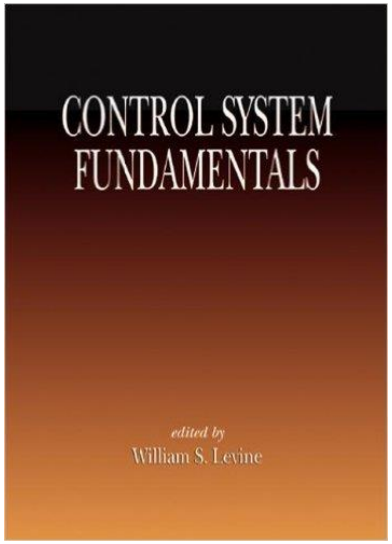
None of the above describes Routh-Hurwitz.
Only Bode plot is used.

Control Theory Text Book

Most of control theory text books
describe Routh-Hurwitz method
for system stability analysis and design !







Research Objective

Our proposal

For

Analysis and design of operational amplifier stability

Use

Routh-Hurwitz stability criterion

↓

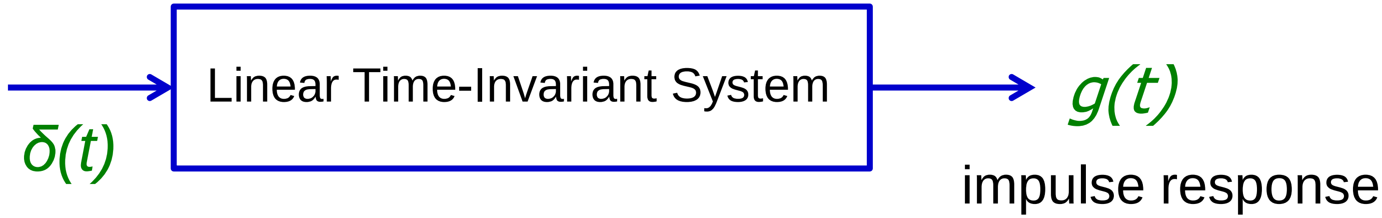
We can obtain

Explicit stability condition for circuit parameters

(which can NOT be obtained only with Bode plot).

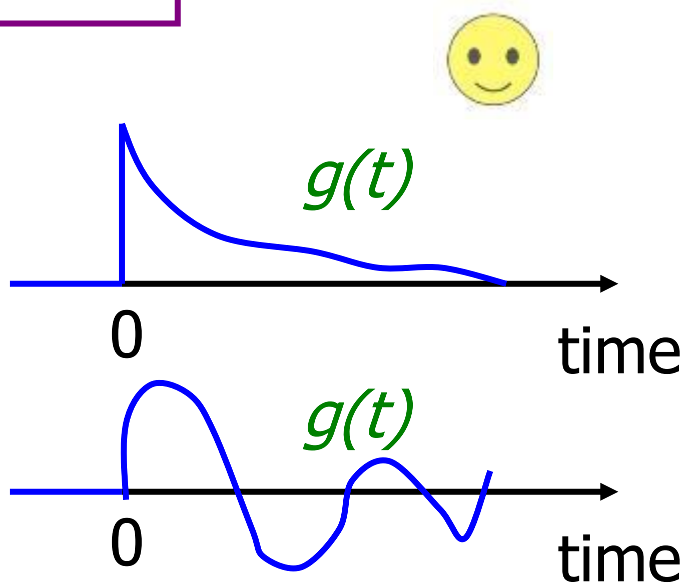
Contents

Stability of Linear Time-Invariant System

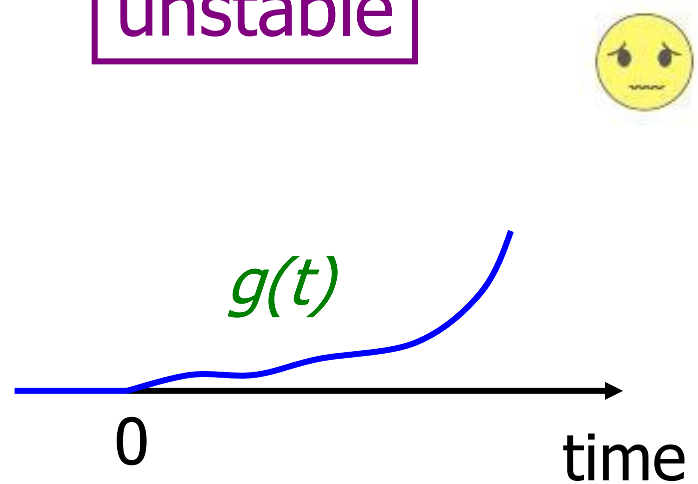


System is stable ⟺ $\lim_{t\to\infty} g(t) = 0$

Stable



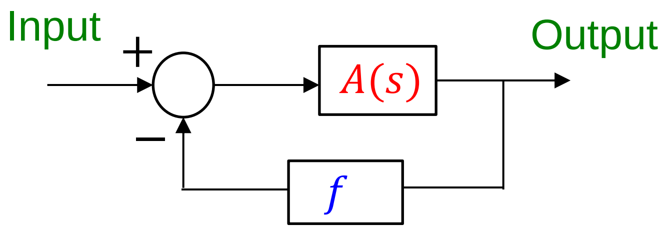
unstable



Stability Criteria of Linear Feedback System

- Problem:

Feedback system is stable or not ?



- Open-loop frequency characteristics of $fA(j\omega)$

➡ Nyquist stability criterion
- Bode plot
- Nyquist plot
- Nicholas plot

- Closed-loop transfer function $\dfrac{A(s)}{1+fA(s)}$

➡ Routh-Hurwitz stability criterion

Contents

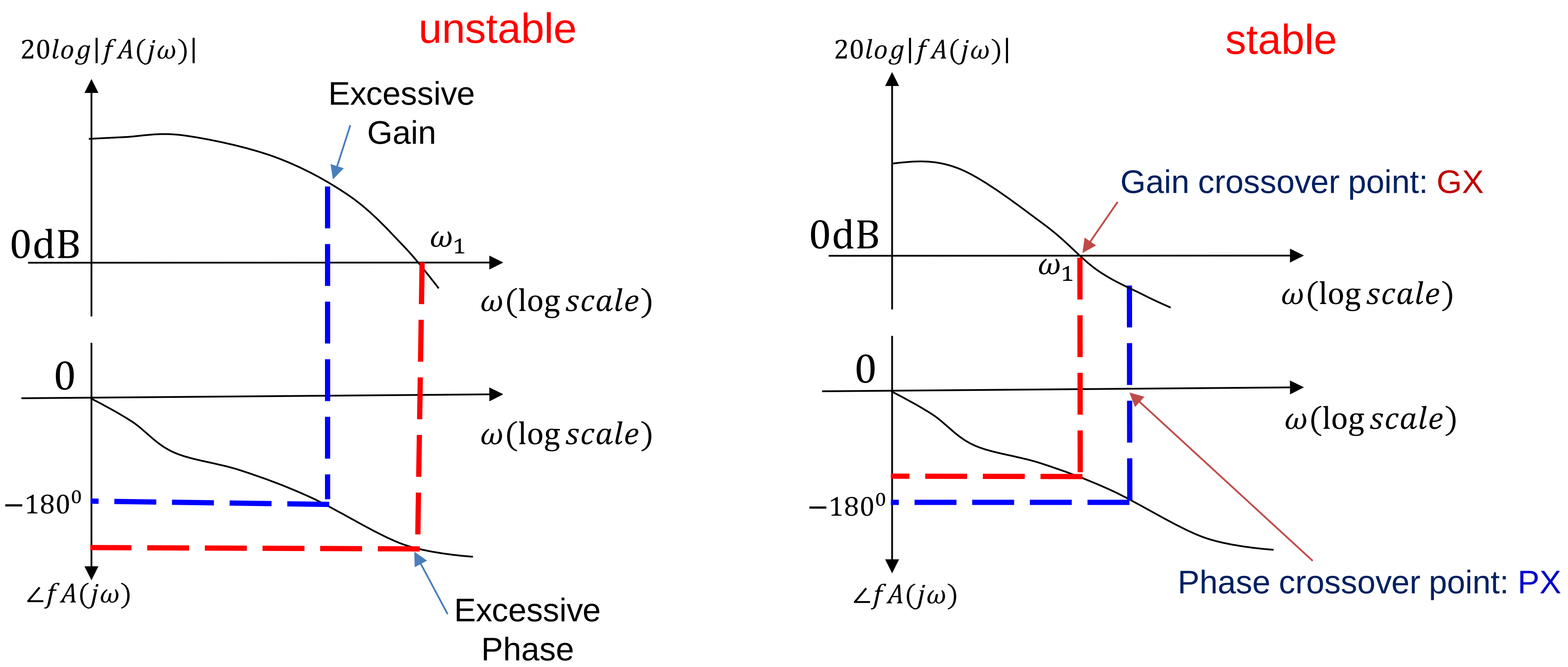
- Research Objective & Background
- Stability Criteria
  - Nyquist Criterion and Bode Plot
  - Routh-Hurwitz Criterion
- Propo od
  - Ex.1: T plifier with ation
  - Ex.2: T plifier with nsation
  - Ex.3: T mplifier w sation
- Discussion & Conclusion

Harry Nyquist
1889-1976 (Sweden)

Hendrik Wade Bode
1905-1982 (蘭)

Bode Plot (Gain & Phase vs Freq.)

Open-loop frequency characteristics of $fA(j\omega)$

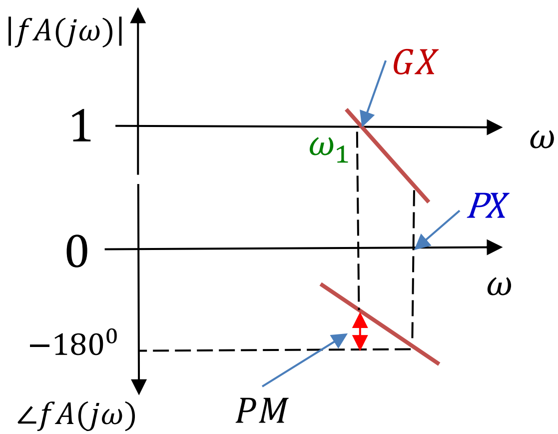


Stable system ⇨ gain crossover GX before phase crossover PX.

Used for frequency characteristics, stability check, gain & phase margins

Phase Margin from Bode Plot

GX precedes $\mathrm{P}X$ ⇒ feedback system is stable.



Greater spacing between GX and $\mathrm{P}X$

⇓

More stable

ω_1: gain crossover frequency

Phase margin : $\mathrm{PM} = 180^0 + \angle fA(\omega = \omega_1)$

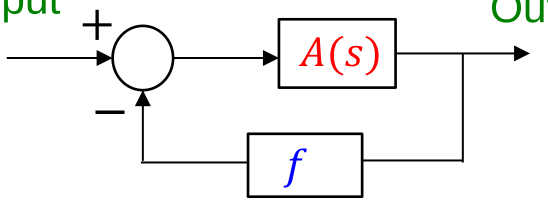
Bode plot is useful,
but it does NOT show explicit stability conditions of circuit parameters.

Contents

Transfer Function and Stability

- Transfer function of closed-loop system

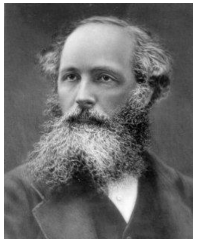
$$G(s) = \frac{A(s)}{1 + fA(s)} = \frac{N(s)}{D(s)}$$



- Suppose

$$N(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0$$

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$


J. Maxwell


A. Stodola

- System is stable if and only if

Maxwell and Stodola found out !!

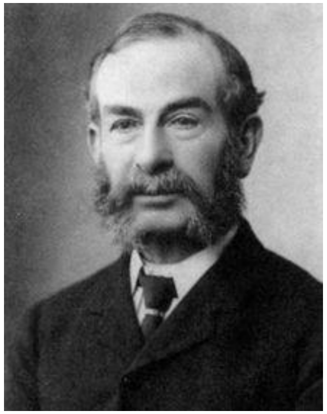
real parts of all the roots s_p of the following are negative:

Characteristic equation $D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$

- To satisfy this, what are the conditions for $a_n, a_{n-1}, \cdots a_1, a_0$?

Routh and Hurwitz solved this problem independently !!

Routh and Hurwitz



Great
Mathematicians !

Edward Routh
1831- 1907（英）

Adolf Hurwitz
1859 - 1919（独）

1876

Routh test

1895

Hurwitz matrix

Very different algorithms,
but later it was proved that both are the same results.

Discover Truth

Routh Stability Criterion

Characteristic equation:

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

Routh table

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$	$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$	b_3	b_4	$\cdots$
s^{n-3}	$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$	$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	a_0				

Sufficient and necessary condition:

(i) $a_i > 0$ for $i = 0,1,\ldots,n$

&

(ii) All values of Routh table's first columns are positive.

Mathematical test

Determine whether given polynomial has all roots in the left-half plane.

Contents

Amplifier Circuit and Small Signal Model

Open-loop transfer function from small signal model

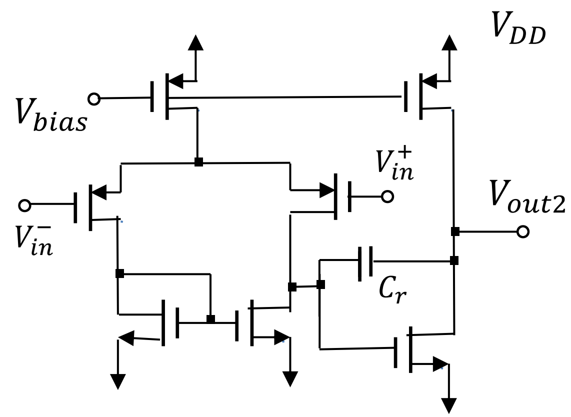
$$A(s) = \frac{v_{out}(s)}{v_{in}(s)} = A_0 \frac{1 + b_1 s}{1 + a_1 s + a_2 s^2}$$

$$b_1 = -\frac{C_r}{G_{m2}}$$

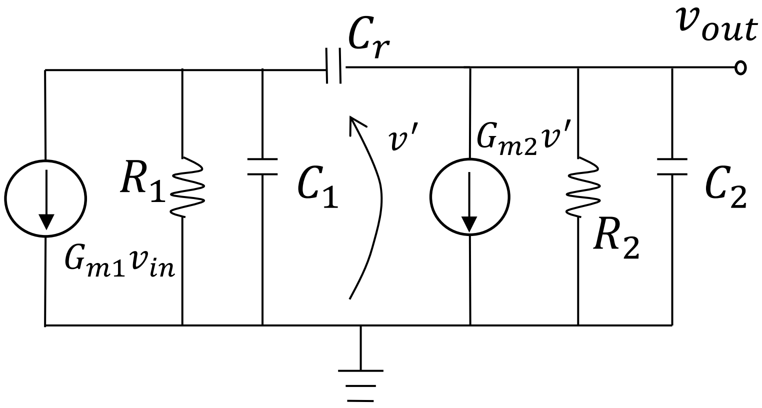
$$A_0 = G_{m1} G_{m2} R_1 R_2$$

$$a_2 = R_1 R_2 C_2 \left[C_1 + \left(1 + \frac{C_1}{C_2}\right) C_r \right]$$

$$a_1 = R_1 C_1 + R_2 C_2 + (R_1 + R_2 + R_1 G_{m2} R_2) C_r$$



Amplifier circuit

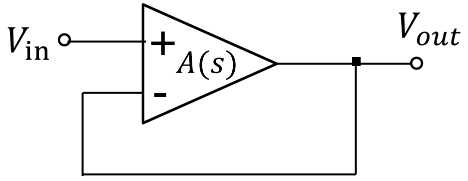


Small signal model

Feedback Configuration

Closed-loop transfer function:

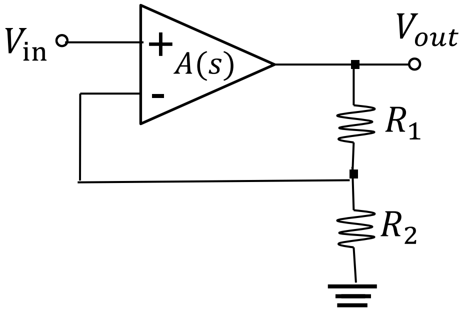
$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{A(s)}{1+fA(s)} = \frac{A_0(1+b_1 s)}{1+fA_0+(a_1+fA_0 b_1)s+a_2 s^2}$$



$f = 1$

Set parameter θ:

$$\theta = a_1 + fA_0 b_1$$



Necessary and sufficient stability condition based on R-H criterion

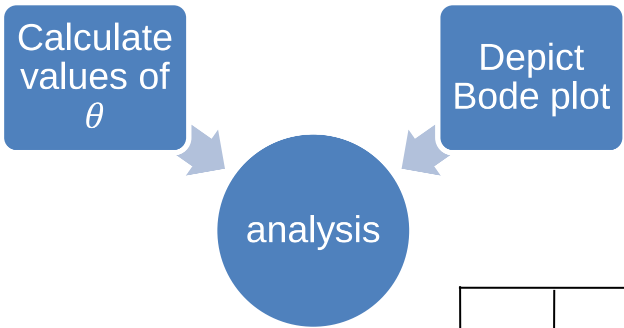
$\Rightarrow$ $\theta > 0$

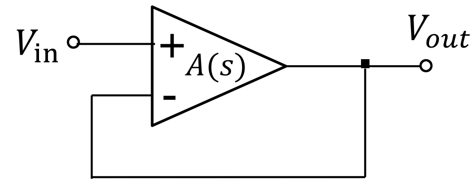
$$f = \frac{R_2}{R_1+R_2}$$

$\Rightarrow$ $R_1C_1 + R_2C_2 + (R_1+R_2)C_r + (G_{m2} - fG_{m1})R_1R_2C_r > 0$

Explicit stability condition of parameters

Verification with SPICE Simulation





voltage follower configuration

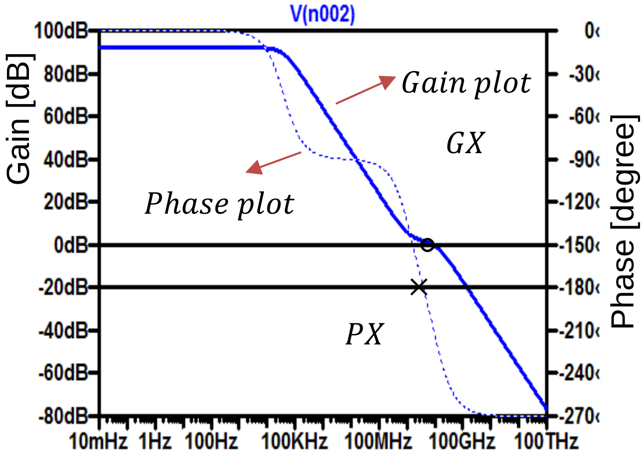
$f = 1$

	Parameter values							R-H criterion	Bode plot
case	R_1	C_1	R_2	C_2	G_{m1}	G_{m2}	C_r	θ	SPICE simulation
(1)	$50k$	$10f$	10k	$0.1p$	0.01	$8m$	$1p$	< 0	unstable
(2)	$50k$	$1f$	10k	$10f$	0.01	$8m$	$0.1p$	< 0	unstable
(3)	$100k$	$100f$	$10k$	$1f$	$9m$	$4m$	$0.1p$	< 0	unstable
(4)	$100k$	$5f$	$90k$	$3f$	$8m$	$7.5m$	$0.9p$	≈ 0	critical stable
(5)	$100k$	$3f$	$50k$	$1f$	$8.5m$	$8m$	$0.5p$	≈ 0	critical stable
(6)	$1meg$	$6f$	$500k$	$0.5f$	$80u$	$70u$	$1f$	≈ 0	critical stable
(7)	$50k$	$10f$	100	$0.1p$	0.01	$8m$	$1p$	> 0	stable
(8)	$100k$	$5f$	$90k$	$3f$	$80u$	$70u$	$0.9p$	> 0	stable
(9)	$150k$	$6f$	$100k$	$1.5f$	$80u$	$70u$	$0.5p$	> 0	stable

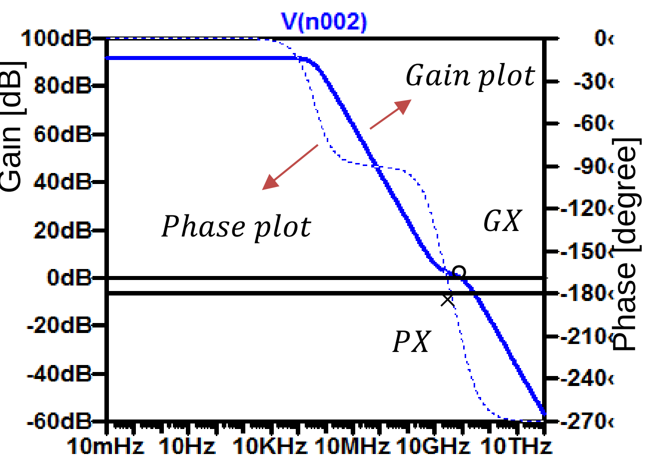
Consistency of Bode Plots and R-H Results

unstable

R-H: $\theta < 0$



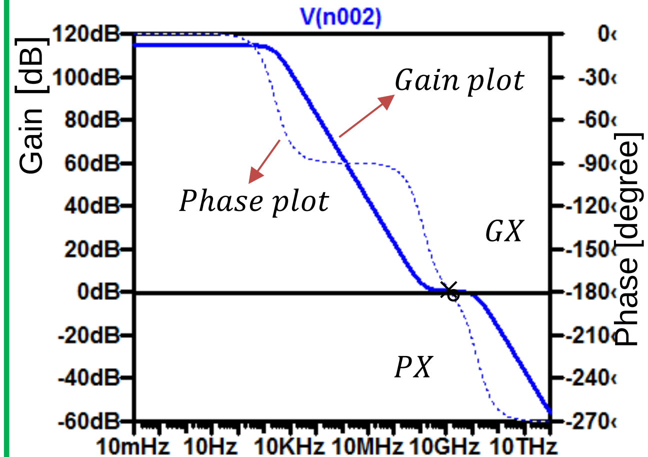
case(1)



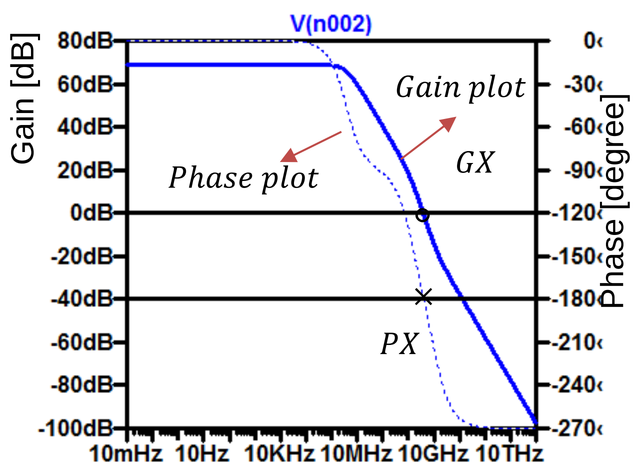
case(2)

critical stable

R-H: $\theta \approx 0$



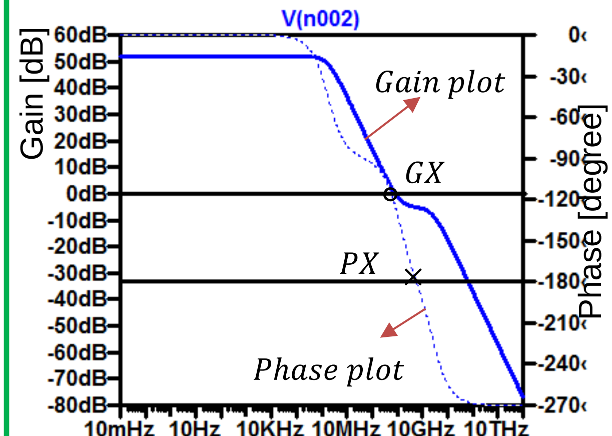
case(4)



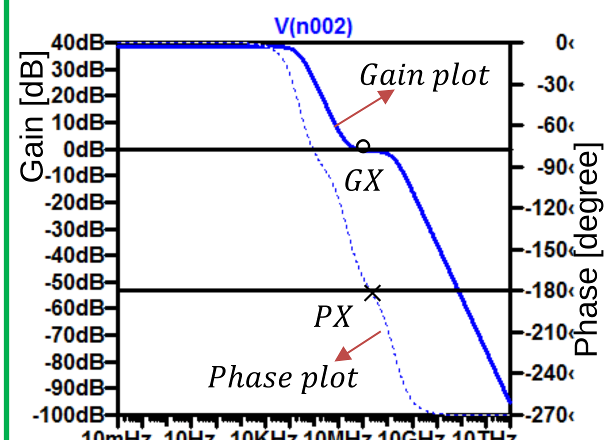
case(6)

stable

R-H: $\theta > 0$

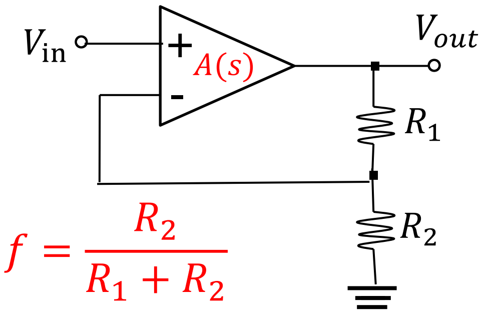


case(7)



case(9)

Consistency of Transient Analysis and R-H Results



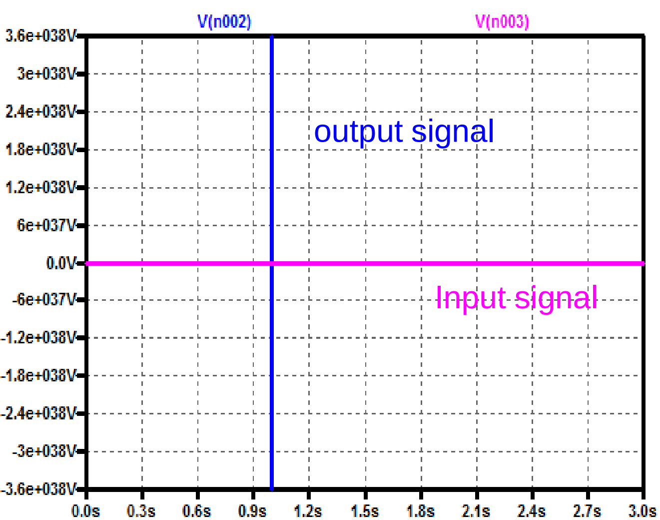
$$f = \frac{R_2}{R_1 + R_2}$$

SPICE simulation

Step response with step input

unstable

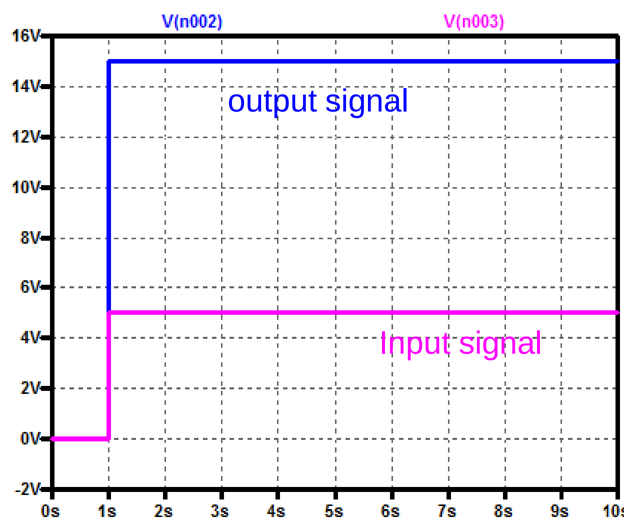
R-H: $\theta < 0$



Case (3)

critical stable

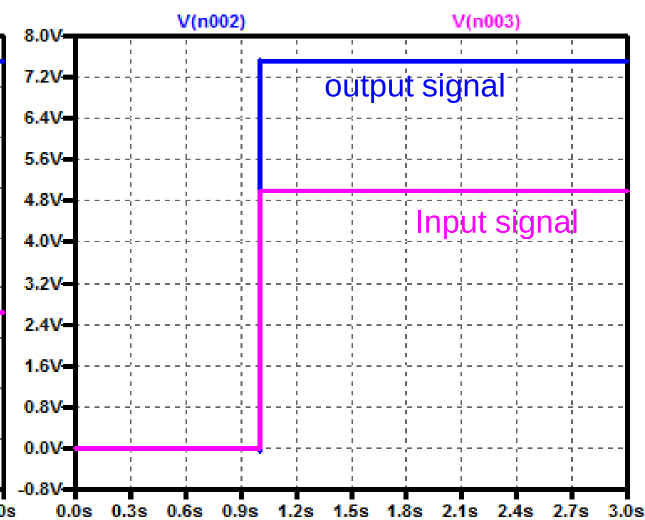
R-H: $\theta \approx 0$



Case (5)

stable

R-H: $\theta > 0$



Case (8)

Contents

Amplifier Circuit and Small Signal Model

Open-loop transfer function:

$$A(s) = \frac{v_{out}(s)}{v_{in}(s)} = A_0 \frac{1 + d_1 s}{1 + a_1 s + a_2 s^2 + a_3 s^3}$$

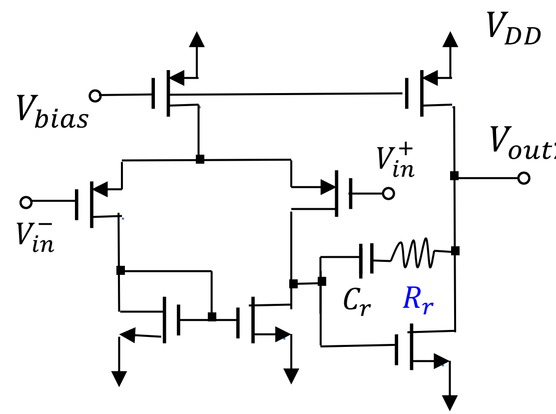
$$A_0 = G_{m1} G_{m2} R_1 R_2$$

$$d_1 = -\left(\frac{C_r}{G_{m2}} - R_r C_r\right)$$

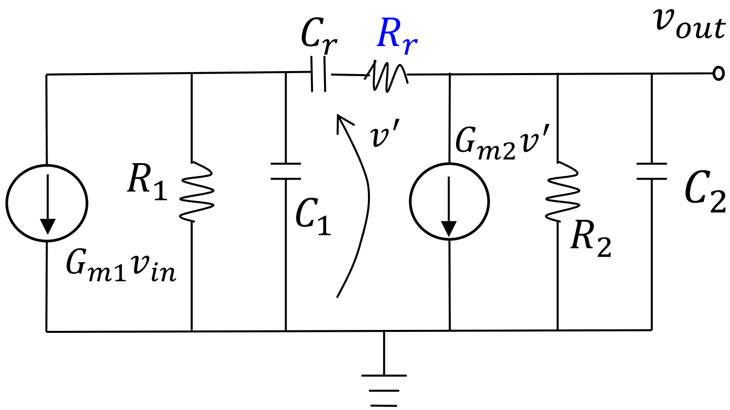
$$a_1 = R_1 C_1 + R_2 C_2 + (R_1 + R_2 + R_r + R_1 R_2 G_{m2}) C_r$$

$$a_2 = R_1 R_2 (C_2 C_r + C_1 C_2 + C_1 C_r) + R_r C_r (R_1 C_1 + R_2 C_2)$$

$$a_3 = R_1 R_2 R_r C_1 C_2 C_r$$



Amplifier circuit

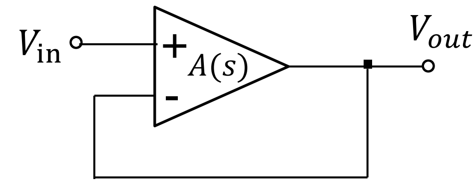


Small signal model

Feedback Configuration

Closed-loop transfer function:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{A(s)}{1+fA(s)} = \frac{A_0(1+d_1 s)}{1+fA_0+(a_1+fA_0 d_1)s+a_2 s^2+a_3 s^3}$$

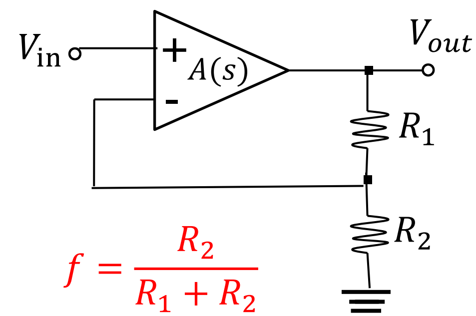


$f = 1$

Set parameter φ:

$$\varphi = a_1 + fA_0 d_1$$

$$= R_1C_1 + R_2C_2 + (R_1+R_2+R_r)C_r + (G_{m2} - fG_{m1} + fG_{m1}G_{m2}R_r)R_1R_2C_r$$



$f = \frac{R_2}{R_1+R_2}$

Necessary and sufficient stability condition based on R-H criterion

⇒ $\varphi > 0$ & b_1 (parameter of Routh stable) > 0

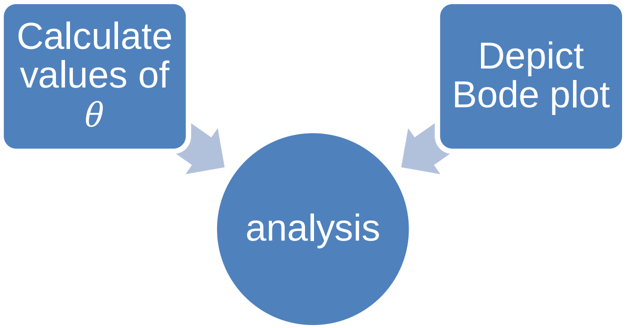
⇒

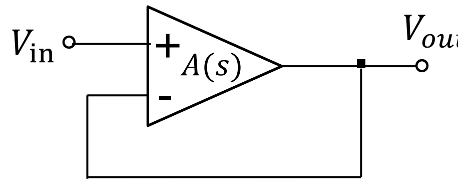
$$R_1C_1 + R_2C_2 + (R_1+R_2+R_r)C_r + (G_{m2} - fG_{m1} + fG_{m1}G_{m2}R_r)R_1R_2C_r > 0$$

$$\frac{(a_1+fA_0d_1)a_2 - a_3(1+fA_0)}{a_2} > 0$$

Explicit stability condition of parameters

Verification with SPICE Simulation





$f = 1$

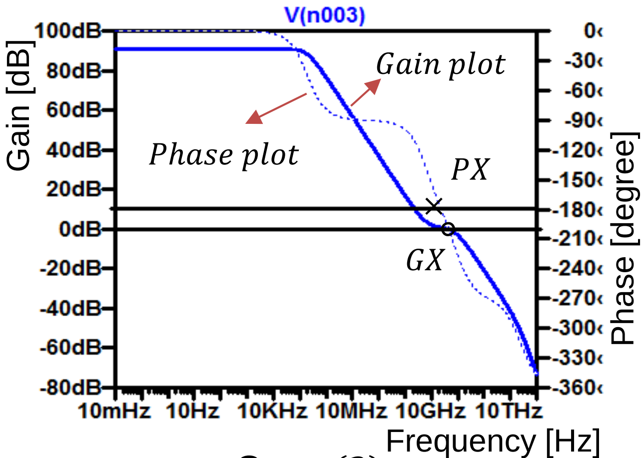
voltage follower configuration

	Parameter values								R-H criterion		Bode plot
case	R_1	C_1	R_2	C_2	G_{m1}	G_{m2}	R_r	C_r	φ	b_1	SPICE simulation
(1)	$115k$	$5f$	$100k$	$80f$	$9m$	$5m$	5	$0.5p$	< 0	< 0	unstable
(2)	$50k$	$5f$	$10k$	$10f$	$9m$	$8m$	2	$0.2p$	< 0	< 0	unstable
(3)	$150k$	$5f$	$100k$	$10f$	$9m$	$8m$	1	$0.8p$	< 0	< 0	unstable
(4)	$110k$	$10f$	$10k$	$3f$	0.01	$8m$	5	$0.5f$	≈ 0	≈ 0	critical
(5)	$115k$	$10f$	$100k$	$3f$	0.01	$8m$	5	$0.5f$	≈ 0	≈ 0	critical
(6)	$150k$	$8f$	$100k$	$50f$	$7m$	$8m$	10	$0.6p$	> 0	> 0	stable
(7)	$100k$	$8f$	$80k$	$50f$	$6m$	$8m$	5	$0.6p$	> 0	> 0	stable
(8)	$200k$	$5f$	$150k$	$10f$	$5m$	$7m$	2.5	$0.6p$	> 0	> 0	stable

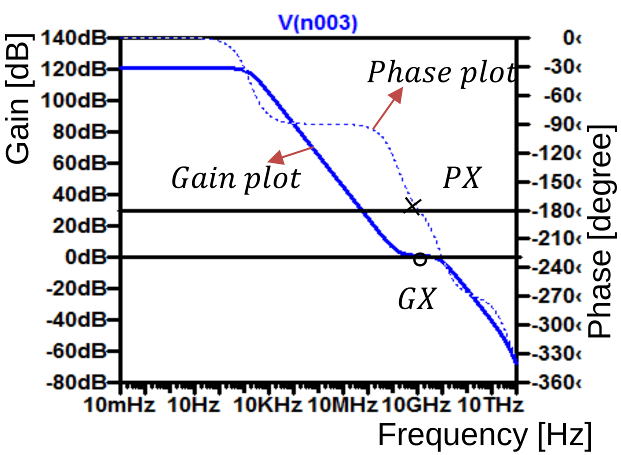
Consistency of Bode Plots and R-H Results

unstable

R-H: $\varphi < 0$



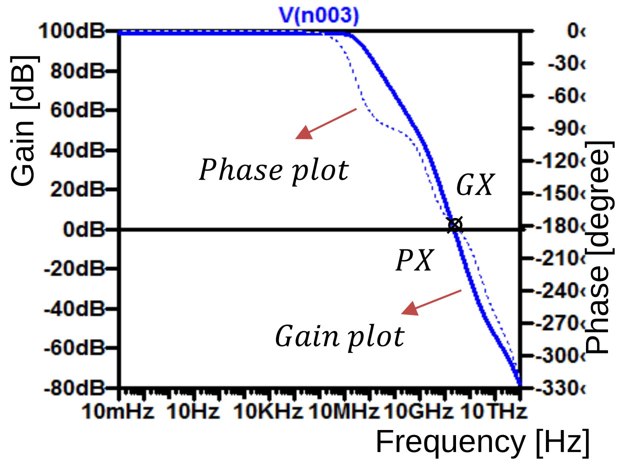
Case (2)



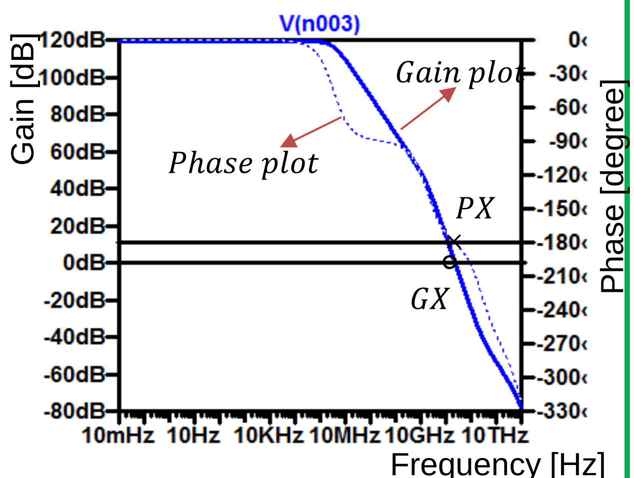
Case (3)

critical stable

R-H: $\varphi \approx 0$



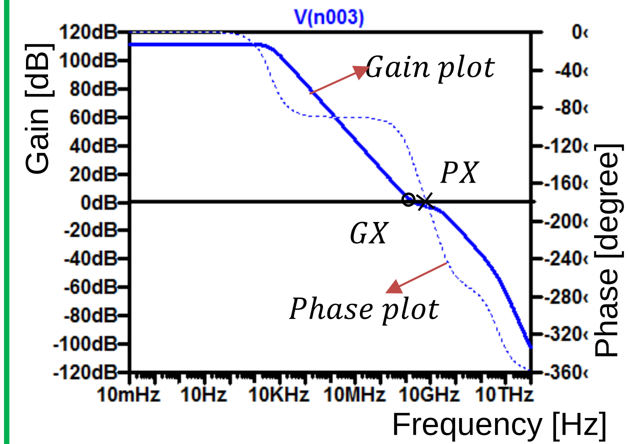
Case (4)



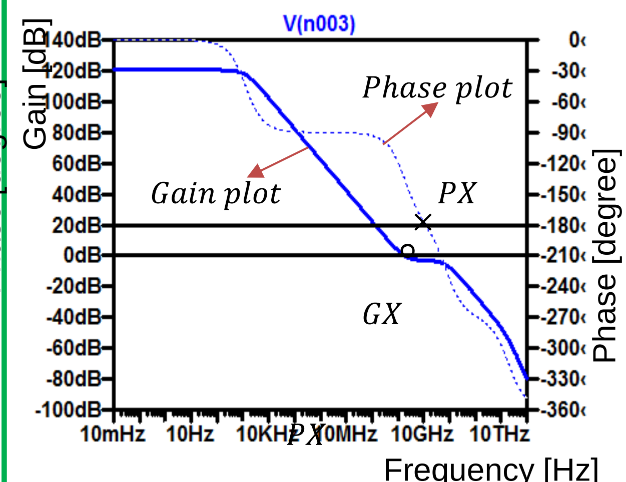
Case (5)

stable

R-H: $\varphi > 0$



Case (7)

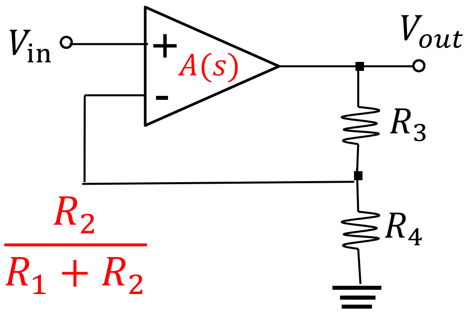


Case (8)

Consistency of Transient Analysis and R-H Results

Linear feedback system:

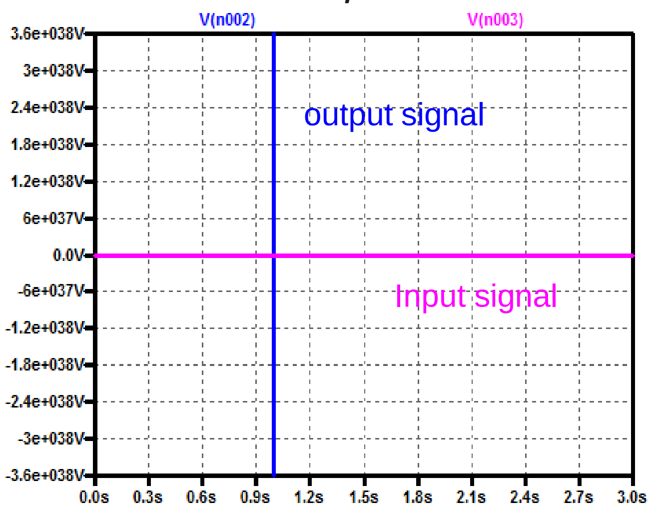
SPICE simulation



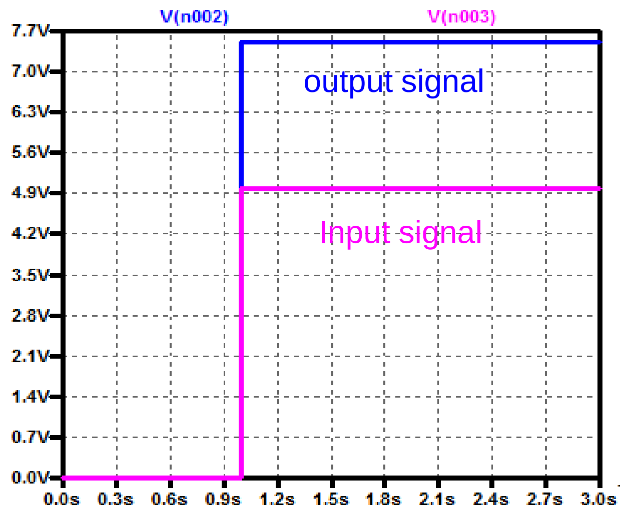
$$f = \frac{R_2}{R_1 + R_2}$$

Step response with step input

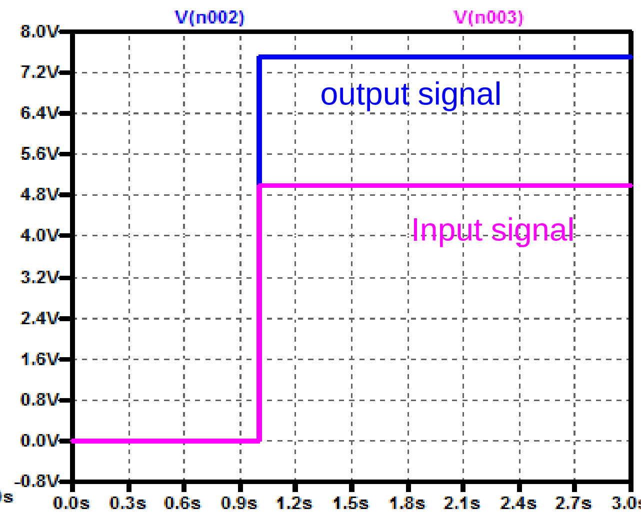
unstable	critical stable	stable
R-H: $\varphi < 0$	R-H: $\varphi \approx 0$	R-H: $\varphi > 0$



Case (1)



Case (5)

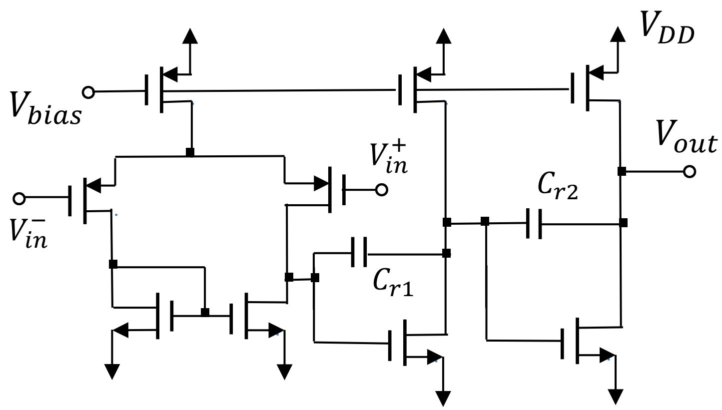


Case (8)

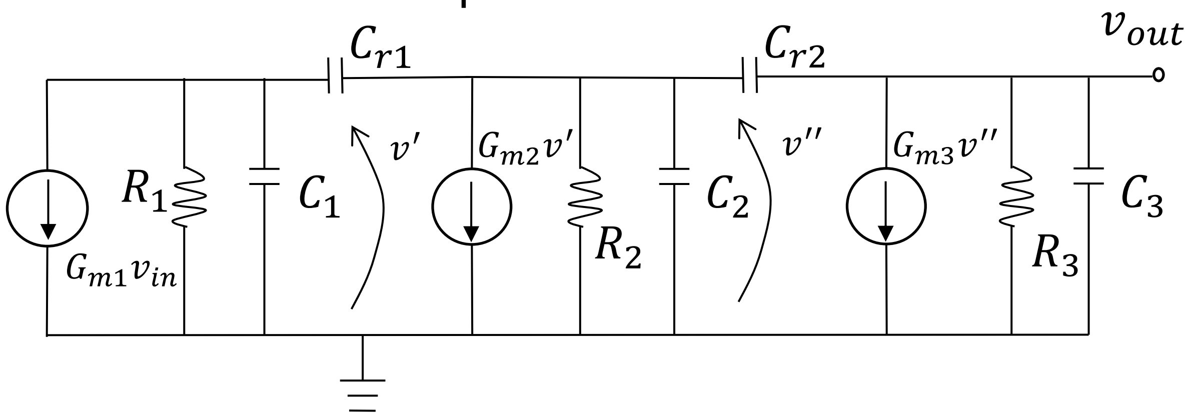
Contents

- Research Objective & Background
- Stability Criteria
  - Nyquist Criterion and Bode Plot
  - Routh-Hurwitz Criterion
- Proposed Method
  - Ex.1: Two-stage amplifier with C compensation
  - Ex.2: Two-stage amplifier with C, R compensation
  - Ex.3: Three-stage amplifier with C compensation
- Discussion & Conclusion

Three-stage Amplifier (3 poles)



Amplifier circuit



Small signal model

Proposed method can be applied in a similar manner.

Contents

Discussion of Proposed Method

Depict small signal equivalent circuit of amplifier

↓

Derive open-loop transfer function

↓

Derive closed-loop transfer function
& obtain characteristics equation

↓

Apply R-H stability criterion
& obtain explicit stability condition

Especially effective for

Multi-stage opamp (high-order system)

Limitation

Explicit transfer function with polynomials of s has to be derived.

Conclusion

- Proposal of Routh-Hurwitz method usage for analysis and design of operational amplifier stability

- Explicit circuit parameter conditions can be obtained for feedback stability.

- Consistency with Bode plot method has been confirmed with SPICE simulation.

⇓

Proposed method can be used
with conventional Bode plot method.

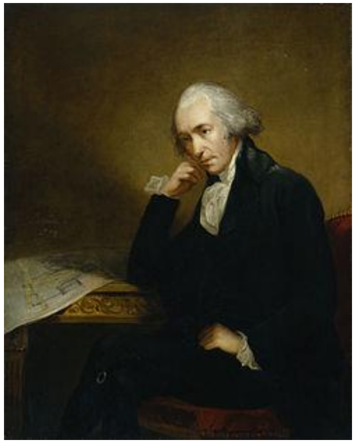
Future work:

Relationship: θ or φ with gain and phase margins

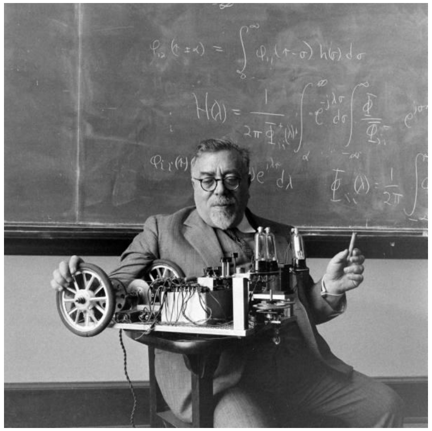
Final Statement

- Control theory is
  theoretical basis of analog circuit design.

- “Feedback” is the most important concept there.



James Watt
1736 - 1819



Nobert Wiener
1894 - 1964

Harold Black
1898-1983

John Ragazzini
1912-1988

Acknowledgements

The authors would like to thank
Prof. Toshiyuki Kitamori
Prof. Hiroshi Tanimoto and
Dr. Yuji Gendai
for stimulating and valuable comments.

Thank you

for your kind attention!

Kobayashi
Laboratory