Analysis and Design of Operational Amplifier Stability Based on Routh-Hurwitz Method

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Contents

● Research Objective & Background

● Stability Criteria
  - Nyquist Criterion and Bode Plot
  - Routh-Hurwitz Criterion

● Proposed Method
  Ex.1: Two-stage amplifier with C compensation
  Ex.2: Two-stage amplifier with C, R compensation
  Ex.3: Three-stage amplifier with C compensation

● Discussion & Conclusion
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● Discussion & Conclusion
Research Background (Stability Theory)

● **Electronic Circuit Design Field**
  - Bode plot (>90% frequently used)
  - Nyquist plot (源代裕治, 電子回路研究会 2015年7月)

● **Control Theory Field**
  - Bode plot
  - Nyquist plot
  - Nicholas plot
  - **Routh-Hurwitz stability criterion**
    - Very popular in control theory field
      but rarely seen in electronic circuit books/papers
  - Lyapunov function method
    :
We were NOT able to find out any electronic circuit text book which describes Routh-Hurwitz method for operational amplifier stability analysis and design!

None of the above describes Routh-Hurwitz. Only Bode plot is used.
Most of control theory text books describe **Routh-Hurwitz** method for system stability analysis and design!
Research Objective

Our proposal

For
Analysis and design of operational amplifier stability

Use
Routh-Hurwitz stability criterion

We can obtain
Explicit stability condition for circuit parameters
(which can NOT be obtained only with Bode plot).
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Stability of Linear Time-Invariant System

\[ \lim_{t \to \infty} g(t) = 0 \]

System is stable \iff \[ g(t) \] is bounded for all time. If \[ g(t) \] grows without bound, the system is unstable.

Graphs illustrate the difference between stable and unstable systems.
Stability Criteria of Linear Feedback System

Problem:
Feedback system is stable or not?

Open-loop frequency characteristics of $fA(j\omega)$

- Nyquist stability criterion

Closed-loop transfer function $\frac{A(s)}{1 + fA(s)}$

- Routh-Hurwitz stability criterion

Diagram:

- Input
- $A(s)$
- $f$
- Output
- Bode plot
  - Nyquist plot
  - Nicholas plot
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Harry Nyquist
1889-1976 (Sweden)

Hendrik Wade Bode
1905-1982 (荷兰)
Bode Plot (Gain & Phase vs Freq.)

Open-loop frequency characteristics of $fA(j\omega)$

Stable system: gain crossover $GX$ before phase crossover $PX$.

Used for frequency characteristics, stability check, gain & phase margins.
Phase Margin from Bode Plot

$GX$ precedes $PX \quad \rightarrow \text{feedback system is stable.}$

Greater spacing between $GX$ and $PX$

More stable

$\omega_1$: gain crossover frequency

Phase margin: $PM = 180^0 + \angle fA(\omega = \omega_1)$

Bode plot is useful, but it does NOT show explicit stability conditions of circuit parameters.
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Transfer Function and Stability

- Transfer function of closed-loop system

\[
G(s) = \frac{A(s)}{1 + fA(s)} = \frac{N(s)}{D(s)}
\]

- Suppose

\[
N(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0
\]
\[
D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0
\]

- System is stable if and only if

Maxwell and Stodola found out !!

real parts of all the roots \( s_p \) of the following are negative:

Characteristic equation

\[
D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0
\]

- To satisfy this, what are the conditions for \( a_n, a_{n-1}, \cdots, a_1, a_0 \)?

Routh and Hurwitz solved this problem independently !!
Routh and Hurwitz

Great Mathematicians!

Edward Routh
1831-1907 (英)

1876
Routh test

Adolf Hurwitz
1859-1919 (独)

1895
Hurwitz matrix

Very different algorithms,
but later it was proved that both are the same results.

Discover Truth
Routh Stability Criterion

Characteristic equation:

\[ D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \]

Sufficient and necessary condition:

(i) \( a_i > 0 \) for \( i = 0, 1, \ldots, n \) 

&

(ii) All values of Routh table’s first columns are positive.

Routh table

<table>
<thead>
<tr>
<th>( S^n )</th>
<th>( a_n )</th>
<th>( a_{n-2} )</th>
<th>( a_{n-4} )</th>
<th>( a_{n-6} )</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S^{n-1} )</td>
<td>( a_{n-1} )</td>
<td>( a_{n-3} )</td>
<td>( a_{n-5} )</td>
<td>( a_{n-7} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( S^{n-2} )</td>
<td>( b_1 = \frac{a_{n-1} a_{n-2} - a_n a_{n-3}}{a_{n-1}} )</td>
<td>( b_2 = \frac{a_{n-1} a_{n-4} - a_n a_{n-5}}{a_{n-1}} )</td>
<td>( b_3 )</td>
<td>( b_4 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( S^{n-3} )</td>
<td>( c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1} )</td>
<td>( c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1} )</td>
<td>( c_3 )</td>
<td>( c_4 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( S^0 )</td>
<td>( a_0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mathematical test

Determine whether given polynomial has all roots in the left-half plane.
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- Discussion & Conclusion
Amplifier Circuit and Small Signal Model

Open-loop transfer function from small signal model

\[
A(s) = \frac{v_{out}(s)}{v_{in}(s)} = A_0 \frac{1 + b_1 s}{1 + a_1 s + a_2 s^2}
\]

\[
b_1 = -\frac{C_r}{G_{m2}}
\]

\[
A_0 = G_{m1} G_{m2} R_1 R_2
\]

\[
a_2 = R_1 R_2 C_2 \left[ C_1 + \left(1 + \frac{C_1}{C_2}\right) C_r \right]
\]

\[
a_1 = R_1 C_1 + R_2 C_2 + (R_1 + R_2 + R_1 G_{m2} R_2) C_r
\]

Amplifier circuit

Small signal model
Feedback Configuration

Closed-loop transfer function:

\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{A(s)}{1 + fA(s)} = \frac{A_0(1 + b_1 s)}{1 + fA_0 + (a_1 + fA_0 b_1)s + a_2 s^2}
\]

Set parameter \( \theta \):

\[
\theta = a_1 + fA_0 b_1
\]

Necessary and sufficient stability condition based on R-H criterion

\[ \theta > 0 \]

\[ R_1 C_1 + R_2 C_2 + (R_1 + R_2) C_r + (G_{m2} - f G_{m1}) R_1 R_2 C_r > 0 \]

Explicit stability condition of parameters
Verification with SPICE Simulation

![voltage follower configuration diagram]

The voltage follower configuration is given by:

\[ V_{\text{in}} + A(s) \cdot V_{\text{out}} \]

\[ f = 1 \]

### Parameter values

<table>
<thead>
<tr>
<th>case</th>
<th>( R_1 )</th>
<th>( C_1 )</th>
<th>( R_2 )</th>
<th>( C_2 )</th>
<th>( G_{m1} )</th>
<th>( G_{m2} )</th>
<th>( C_r )</th>
<th>( \theta )</th>
<th>R-H criterion</th>
<th>Bode plot</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>50k</td>
<td>10f</td>
<td>10k</td>
<td>0.1p</td>
<td>0.01</td>
<td>8m</td>
<td>1p</td>
<td>&lt; 0</td>
<td>unstable</td>
<td></td>
</tr>
<tr>
<td>(2)</td>
<td>50k</td>
<td>1f</td>
<td>10k</td>
<td>10f</td>
<td>0.01</td>
<td>8m</td>
<td>0.1p</td>
<td>&lt; 0</td>
<td>unstable</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>100k</td>
<td>100f</td>
<td>10k</td>
<td>1f</td>
<td>9m</td>
<td>4m</td>
<td>0.1p</td>
<td>&lt; 0</td>
<td>unstable</td>
<td></td>
</tr>
<tr>
<td>(4)</td>
<td>100k</td>
<td>5f</td>
<td>90k</td>
<td>3f</td>
<td>8m</td>
<td>7.5m</td>
<td>0.9p</td>
<td>( \approx 0 )</td>
<td>critical stable</td>
<td></td>
</tr>
<tr>
<td>(5)</td>
<td>100k</td>
<td>3f</td>
<td>50k</td>
<td>1f</td>
<td>8.5m</td>
<td>8m</td>
<td>0.5p</td>
<td>( \approx 0 )</td>
<td>critical stable</td>
<td></td>
</tr>
<tr>
<td>(6)</td>
<td>1meg</td>
<td>6f</td>
<td>500k</td>
<td>0.5f</td>
<td>80u</td>
<td>70u</td>
<td>1f</td>
<td>( \approx 0 )</td>
<td>critical stable</td>
<td></td>
</tr>
<tr>
<td>(7)</td>
<td>50k</td>
<td>10f</td>
<td>100</td>
<td>0.1p</td>
<td>0.01</td>
<td>8m</td>
<td>1p</td>
<td>&gt; 0</td>
<td>stable</td>
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</tr>
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<td>100k</td>
<td>5f</td>
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<td>3f</td>
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<td>70u</td>
<td>0.9p</td>
<td>&gt; 0</td>
<td>stable</td>
<td></td>
</tr>
<tr>
<td>(9)</td>
<td>150k</td>
<td>6f</td>
<td>100k</td>
<td>1.5f</td>
<td>80u</td>
<td>70u</td>
<td>0.5p</td>
<td>&gt; 0</td>
<td>stable</td>
<td></td>
</tr>
</tbody>
</table>

### R-H criterion

- \( R-H \) criterion checks the stability of the system.

### Bode plot

- Stability analysis using SPICE simulation.
Consistency of Bode Plots and R-H Results

unstable
R-H: $\theta < 0$

critical stable
R-H: $\theta \approx 0$

stable
R-H: $\theta > 0$

case (1)  Frequency [Hz]

Gain [dB]  Phase [degree]

Gain plot  Phase plot

$V(n002)$

$GX$

$PX$

Frequency [Hz]

case (4)  Frequency [Hz]

Gain [dB]  Phase [degree]

Gain plot  Phase plot

$V(n002)$

$GX$

$PX$

Frequency [Hz]

case (7)  Frequency [Hz]

Gain [dB]  Phase [degree]

Gain plot  Phase plot

$V(n002)$

$GX$

$PX$

Frequency [Hz]

case (2)  Frequency [Hz]

Gain [dB]  Phase [degree]

Gain plot  Phase plot

$V(n002)$

$GX$

$PX$

Frequency [Hz]

case (6)  Frequency [Hz]

Gain [dB]  Phase [degree]

Gain plot  Phase plot

$V(n002)$

$GX$

$PX$

Frequency [Hz]

case (9)  Frequency [Hz]

Gain [dB]  Phase [degree]

Gain plot  Phase plot

$V(n002)$

$GX$

$PX$

Frequency [Hz]
Consistency of Transient Analysis and R-H Results

\[ f = \frac{R_2}{R_1 + R_2} \]

SPICE simulation

Step response with step input

unstable

\[ \text{R-H: } \theta < 0 \]

critical stable

\[ \text{R-H: } \theta \approx 0 \]

stable

\[ \text{R-H: } \theta > 0 \]
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Amplifier Circuit and Small Signal Model

Open-loop transfer function:

\[ A(s) = \frac{v_{out}(s)}{v_{in}(s)} = A_0 \frac{1 + d_1 s}{1 + a_1 s + a_2 s^2 + a_3 s^3} \]

\[ A_0 = G_{m1}G_{m2}R_1R_2 \]

\[ d_1 = -\left(\frac{c_r}{G_{m2}} - R_r C_r\right) \]

\[ a_1 = R_1 C_1 + R_2 C_2 + (R_1 + R_2 + R_r + R_1 R_2 G_{m2}) C_r \]

\[ a_2 = R_1 R_2 (C_2 C_r + C_1 C_2 + C_1 C_r) + R_r C_r (R_1 C_1 + R_2 C_2) \]

\[ a_3 = R_1 R_2 R_r C_1 C_2 C_r \]
Feedback Configuration

Closed-loop transfer function:

\[
\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{A(s)}{1 + fA(s)} = \frac{A_0(1 + d_1s)}{1 + fA_0 + (a_1 + fA_0d_1)s + a_2s^2 + a_3s^3}
\]

Set parameter \( \varphi \):

\[
\varphi = a_1 + fA_0d_1 = R_1C_1 + R_2C_2 + (R_1 + R_2 + R_r)C_r + (G_{m2} - fG_{m1} + fG_{m1}G_{m2}R_r)R_1R_2C_r
\]

Necessary and sufficient stability condition based on R-H criterion

\( \varphi > 0 \) \& \( b_1 \) (parameter of Routh stable) \( > 0 \)

\[
R_1C_1 + R_2C_2 + (R_1 + R_2 + R_r)C_r + (G_{m2} - fG_{m1} + fG_{m1}G_{m2}R_r)R_1R_2C_r > 0
\]

\[
\frac{(a_1 + fA_0d_1)a_2 - a_3(1 + fA_0)}{a_2} > 0
\]

Explicit stability condition of parameters
Verification with SPICE Simulation

- Voltage follower configuration

\[ V_{in} \rightarrow A(s) \rightarrow V_{out} \]

\[ f = 1 \]

<table>
<thead>
<tr>
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<th>( C_1 )</th>
<th>( R_2 )</th>
<th>( C_2 )</th>
<th>( G_{m1} )</th>
<th>( G_{m2} )</th>
<th>( R_r )</th>
<th>( C_r )</th>
<th>( \varphi )</th>
<th>( b_1 )</th>
<th>Bode plot</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>115k</td>
<td>5f</td>
<td>100k</td>
<td>80f</td>
<td>9m</td>
<td>5m</td>
<td>5</td>
<td>0.5p</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>unstable</td>
</tr>
<tr>
<td>(2)</td>
<td>50k</td>
<td>5f</td>
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<td>9m</td>
<td>8m</td>
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<td>&lt; 0</td>
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</tr>
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<td>150k</td>
<td>5f</td>
<td>100k</td>
<td>10f</td>
<td>9m</td>
<td>8m</td>
<td>1</td>
<td>0.8p</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>unstable</td>
</tr>
<tr>
<td>(4)</td>
<td>110k</td>
<td>10f</td>
<td>10k</td>
<td>3f</td>
<td>0.01</td>
<td>8m</td>
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<td>0.5f</td>
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<td>8f</td>
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<td>6m</td>
<td>8m</td>
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<td>10f</td>
<td>5m</td>
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<td>0.6p</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>stable</td>
</tr>
</tbody>
</table>
Consistency of Bode Plots and R-H Results

unstable
R-H: $\phi < 0$

Case (2)

critical stable
R-H: $\phi \approx 0$

Case (4)

Case (7)

stable
R-H: $\phi > 0$

Case (3)

Case (5)

Case (8)
Consistency of Transient Analysis and R-H Results

Linear feedback system:

\[ f = \frac{R_2}{R_1 + R_2} \]

SPICE simulation

Step response with step input

R-H: \( \varphi < 0 \) (unstable)

R-H: \( \varphi \approx 0 \) (critical stable)

R-H: \( \varphi > 0 \) (stable)
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- Discussion & Conclusion
Three-stage Amplifier (3 poles)

Amplifier circuit

Small signal model

Proposed method can be applied in a similar manner.
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Discussion of Proposed Method

Depict small signal equivalent circuit of amplifier

Derive open-loop transfer function

Derive closed-loop transfer function & obtain characteristics equation

Apply R-H stability criterion & obtain explicit stability condition

Especially effective for

Multi-stage opamp (high-order system)

Limitation

Explicit transfer function with polynomials of $s$ has to be derived.
Conclusion

● Proposal of Routh-Hurwitz method usage for analysis and design of operational amplifier stability

● Explicit circuit parameter conditions can be obtained for feedback stability.

● Consistency with Bode plot method has been confirmed with SPICE simulation.

Proposed method can be used with conventional Bode plot method.

Future work:

Relationship: $\theta$ or $\varphi$ with gain and phase margins
Final Statement

● Control theory is the theoretical basis of analog circuit design.

● “Feedback” is the most important concept there.

James Watt 1736 - 1819
Nobert Wiener 1894 - 1964
Harold Black 1898-1983
John Ragazzini 1912-1988
Acknowledgements

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Thank you for your kind attention!