Design of Operational Amplifier Stability and Phase Margin Using Routh-Hurwitz Method

王建龍　築地伸和　小林春夫

群馬大学大学院　理工学府電子情報部門

群馬大学
GUNMA UNIVERSITY

Contents

Contents

- Research Objective & Background
- Stability Criteria
  - Nyquist Criterion
  - Routh-Hurwitz Criterion
- Relationship between Routh-Hurwitz criterion parameter with phase margin
- Simulation Verification
- Discussion & Conclusion

Research Background (Stability Theory)

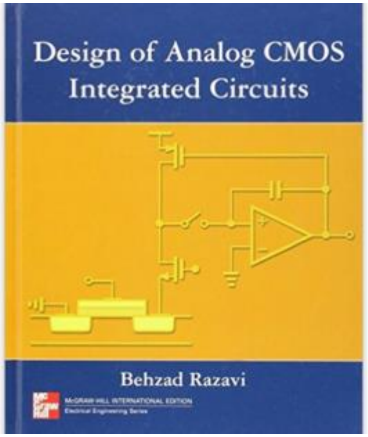
- **Electronic Circuit Design Field**
  - Bode plot (>90% frequently used)
  - Nyquist plot

- **Control Theory Field**
  - Bode plot
  - Nyquist plot
  - Nicholas plot
  - Routh-Hurwitz stability criterion
    ⇨ Very popular in control theory field
    but rarely seen in electronic circuit books/papers
  - Lyapunov function method

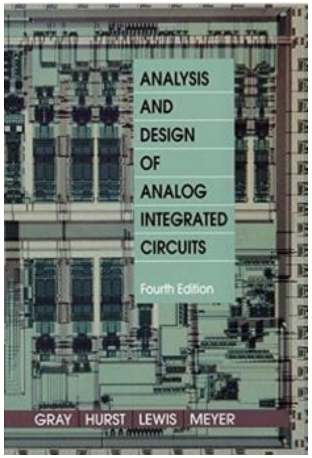
:

Electronic Circuit Text Book

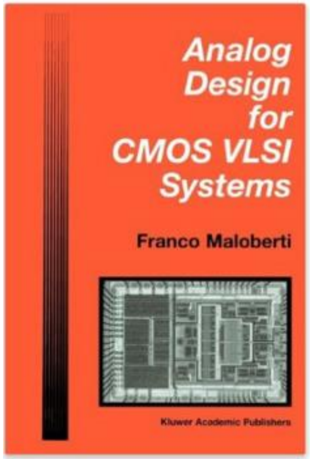
We were NOT able to find out any electronic circuit text book which describes Routh-Hurwitz method
for operational amplifier stability analysis and design !



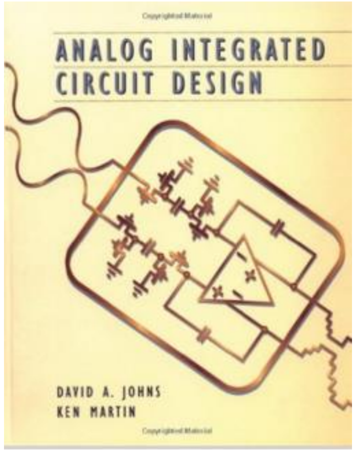
Razavi



Gray



Maloberti

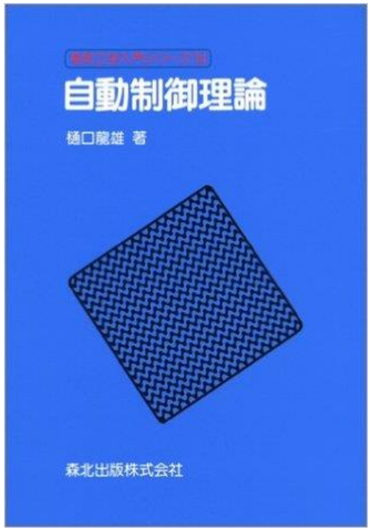


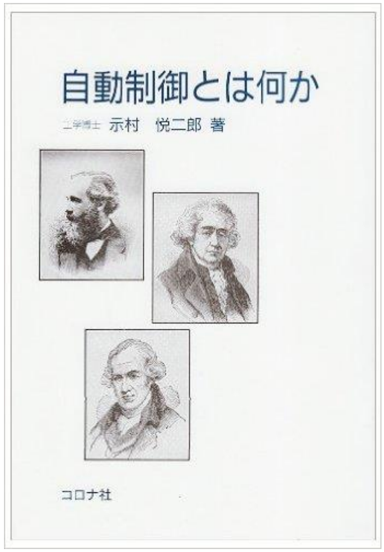
Martin

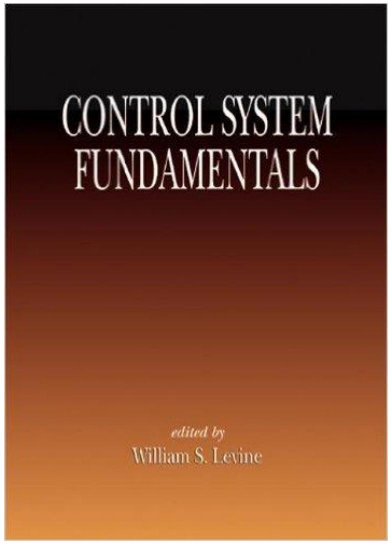
None of the above describes Routh-Hurwitz.
Only Bode plot is used.

Control Theory Text Book

Most of control theory text books
describe Routh-Hurwitz method
for system stability analysis and design !







Research Objective

Our proposal

For

Analysis and design of operational amplifier stability

Use

Routh-Hurwitz stability criterion

↓

We can obtain

- Explicit stability condition for circuit parameters
  (which can NOT be obtained only with Bode plot)
- Monotonic relationship between R-H criterion parameter with phase margin

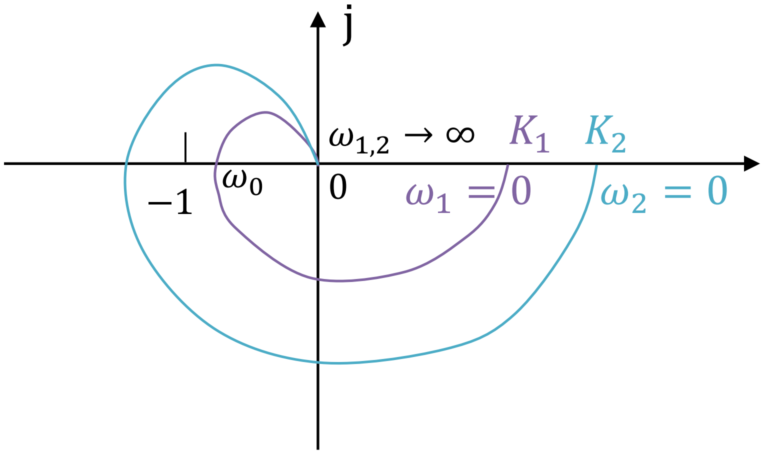
Contents

- Research Objective & Background
- Stability Criteria
  - Nyquist Criterion (Nyquist plot, Bode plot)
  - Routh-Hurwitz Criterion
- Relationship between Routh-Hurwitz criterion parameter with phase margin
- Simulation Verification
- Discussion & Conclusion

Nyquist plot

- Open-loop frequency characteristic

  ⇒ Closed-loop stability



Nyquist plot of open-loop system

- Necessary and sufficient condition :

  When $\omega = 0 \to \infty$, $N = P - Z$

N : number, Nyquist plot anti-clockwise encircle point (-1,j0).

P: number, positive roots of open-loop characteristic equation.

Z: number, positive roots of closed-loop characteristic equation.

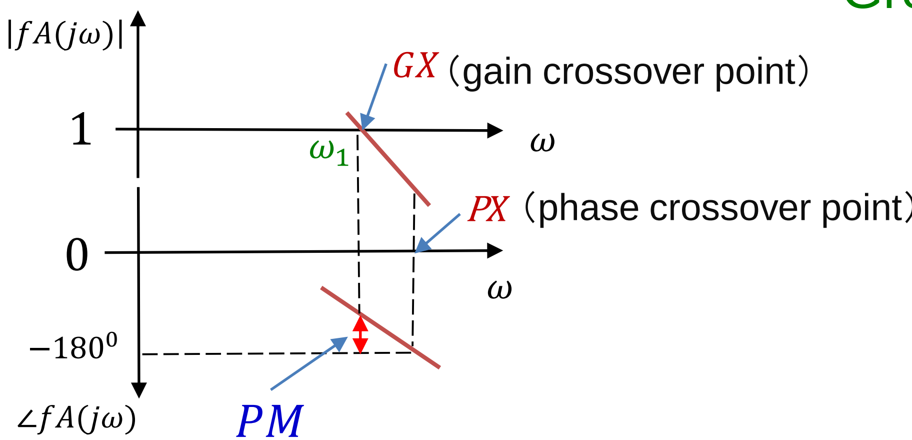
- If the open-loop system is stable(P=0), the Nyquist plot mustn't encircle the point (-1,j0).

⇓

$$\angle G_{open}(j\omega_0) = -\pi, \left|G_{open}(j\omega_0)\right| < 1$$

Phase Margin from Bode Plot

GX precedes PX ⇒ Feedback system is stable



Greater spacing between GX and PX

⇓

More stable

ω_1: gain crossover frequency

Phase margin : $\mathrm{PM} = 180^0 + \angle fA(\omega = \omega_1)$

> Bode plot is useful,
> but it does NOT show explicit stability conditions of circuit parameters.

Contents

Routh Stability Criterion

Characteristic equation:

$$D(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0$$

Sufficient and necessary condition:

(i) $\alpha_i > 0 \;\; \text{for } i = 0,1,\ldots,n$

&

(ii) All values of Routh table's first columns are positive.

Routh table

s^n	α_n	α_{n-2}	α_{n-4}	α_{n-6}	$\cdots$
s^{n-1}	α_{n-1}	α_{n-3}	α_{n-5}	α_{n-7}	$\cdots$
s^{n-2}	$\beta_1 = \frac{\alpha_{n-1}\alpha_{n-2} - \alpha_n\alpha_{n-3}}{\alpha_{n-1}}$	$\beta_2 = \frac{\alpha_{n-1}\alpha_{n-4} - \alpha_n\alpha_{n-5}}{\alpha_{n-1}}$	β_3	β_4	$\cdots$
s^{n-3}	$\gamma_1 = \frac{\beta_1\alpha_{n-3} - \alpha_{n-1}\beta_2}{\beta_1}$	$\gamma_2 = \frac{\beta_1\alpha_{n-5} - \alpha_{n-1}\beta_3}{\beta_1}$	γ_3	γ_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	α_0				

Mathematical test

Determine whether given polynomial has all roots in the left-half plane.

Contents

Contents

- Research Objective & Background
- Stability Criteria
  - Nyquist Criterion
  - Routh-Hurwitz Criterion
- Relationship between Routh-Hurwitz criterion parameter with phase margin
  - Ex.1: Two-stage amplifier with C compensation
  - Ex.2: Two-stage amplifier with C, R compensation
- Simulation Verification
- Discussion & Conclusion

Amplifier Circuit and Small Signal Model

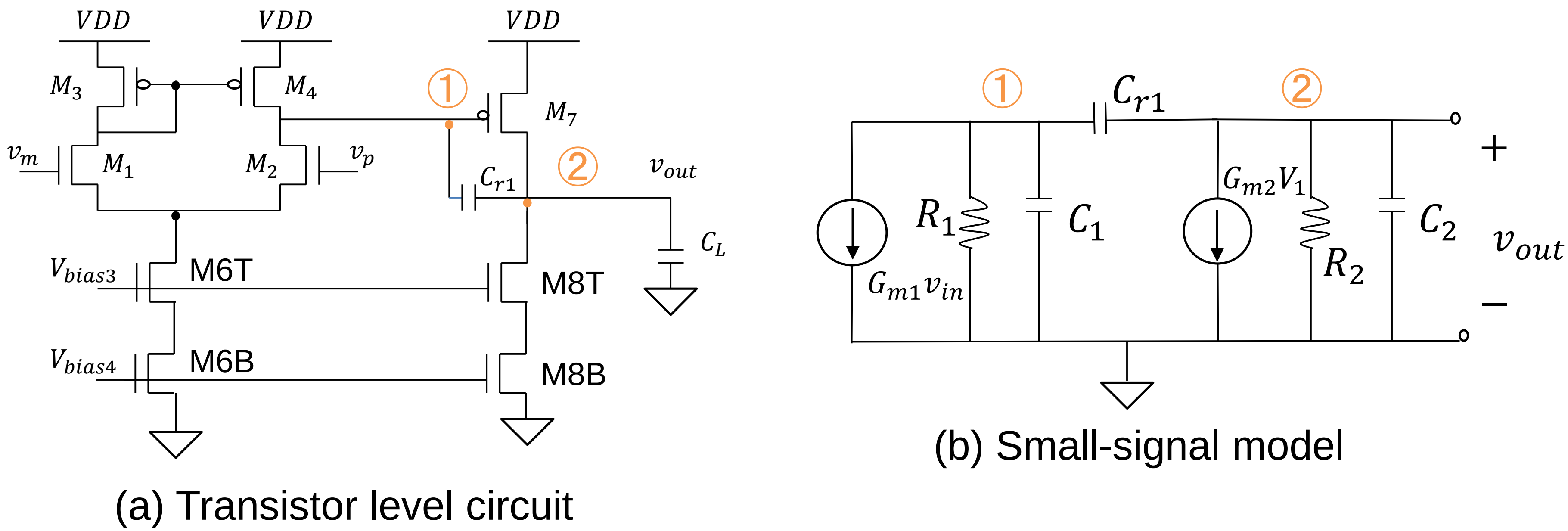


Fig.1 Two-pole amplifier with inter-stage capacitance

Open-loop transfer function from small signal model

$$A(s) = \frac{v_{out}(s)}{v_{in}(s)} = A_0 \frac{1 + b_1 s}{1 + a_1 s + a_2 s^2}$$

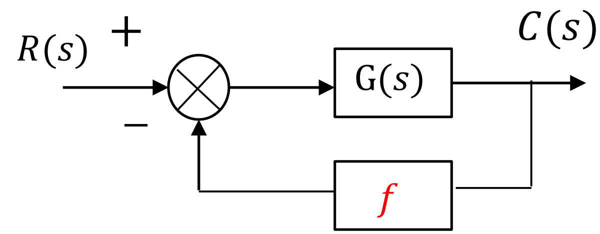
$$b_1 = -\frac{C_r}{G_{m2}} \qquad A_0 = G_{m1} G_{m2} R_1 R_2$$

$$a_1 = R_1 C_1 + R_2 C_2 + (R_1 + R_2 + R_1 G_{m2} R_2) C_r \qquad a_2 = R_1 R_2 (C_1 C_2 + C_1 C_r + C_2 C_r)$$

Routh-Hurwitz method

Closed-loop transfer function:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{A(s)}{1+fA(s)} = \frac{A_0(1+b_1 s)}{1+fA_0+(a_1+fA_0b_1)s+a_2s^2}$$



Closed-loop configuration

Explicit stability condition of parameters:

$$\theta = a_1 + fA_0b_1$$
$$= R_1C_1 + R_2C_2 + (R_1+R_2)C_r + (G_{m2} - fG_{m1})R_1R_2C_r > 0$$

θ: time dimension parameter

Relationship: θ and phase margin

↓

MATLAB

↓

Data fitting

Short-channel CMOS parameters:

$R_1 = r_{on}||r_{op} = 111k\Omega$

$R_2 = r_{op}||R_{ocasn} \approx r_{op} = 333k\Omega$

$G_{m1} = g_{mn} = 100\,uA/V$

$G_{m2} = g_{mp} = 180\,uA/V$

$C_1 = C_{dg4} + C_{dg2} + C_{gs7} = 13.6fF$

$C_2 = C_L + C_{gd8} \approx C_L + 1.56fF$
$= 101.56fF \quad (C_L = 100fF)$

Data Processing by MATLAB

- Data collection: [GM, PM, F_{gm}, F_{pm}]=margin(G)

f=0.01										
C_{r1} [fF]	10	20	30	40	50	60	70	80	90	…
θ [uS]	0.11	0.18	0.25	0.32	0.39	0.46	0.53	0.60	0.67	…
PM [degree]	16	19	22	24	27	29	31	33	34	…
GM [dB]	9.1	7.6	7.0	6.6	6.4	6.3	6.2	6.0	6.0	…
F_{gm} [GHz]	4.5	3.4	2.9	2.6	2.3	2.1	2.0	1.9	1.8	…
F_{pm} [GHz]	2.6	2.1	1.8	1.5	1.4	1.2	1.1	1.0	9.4	…

- Data fitting： p=polyfit(x,y,n) Curve Fitting Tool

Data Fitting Result

Fitted Curve

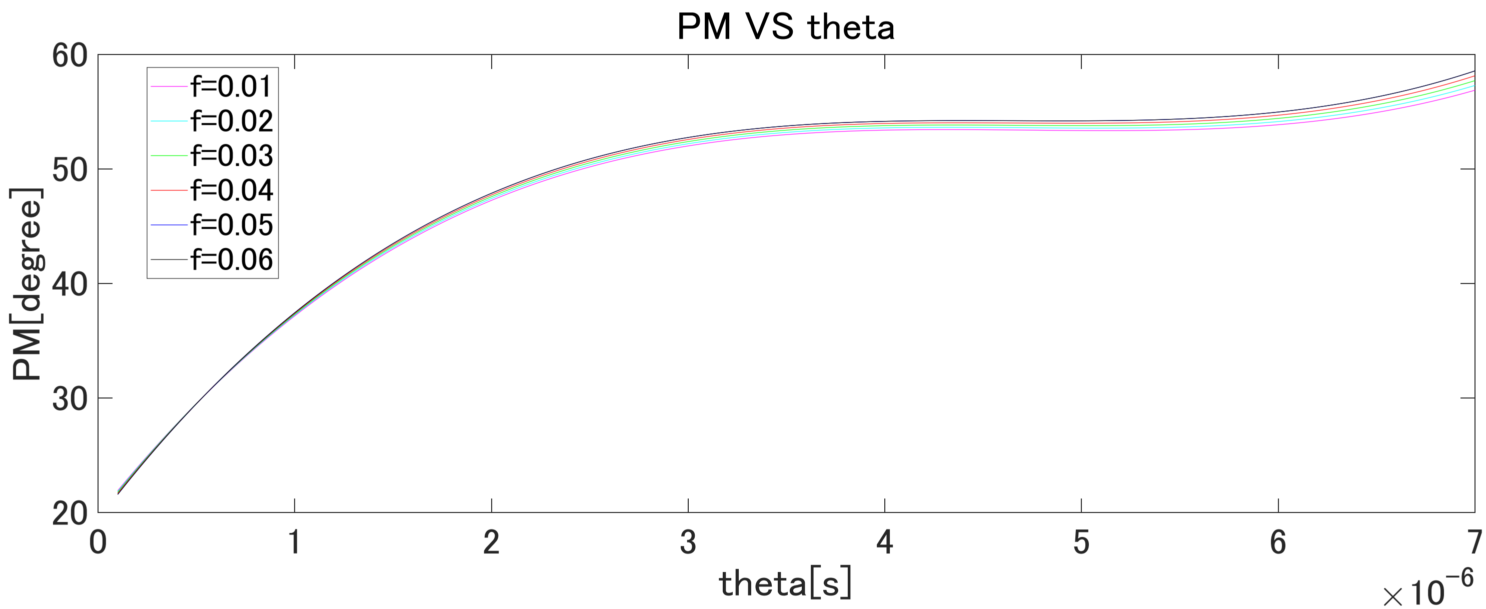
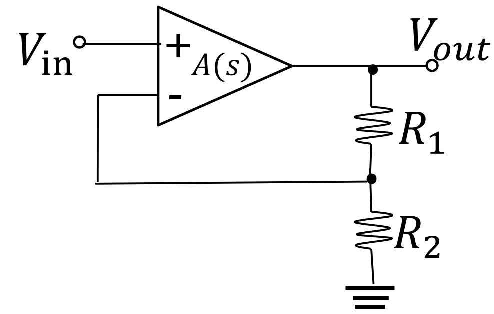


Fig.2 Relationship between PM with parameter θ at variation feedback factor f conditions.

- Monotonic relationship
- Following with the increase of parameter's value
  - ⇨ the phase margin will be increased
  - feedback system will be more stable

f =0.01 Condition



Relation function:

$$PM = f_1(\theta)$$
$$= 2.601e^{28}\theta^5 - 5.616e^{23}\theta^4 + 4.683e^{18}\theta^3 - 1.915e^{13}\theta^2 + 4.076e^{28}\theta + 13.38$$

θ: independent variable
PM: dependent variable

$$f = \frac{R_2}{R_1 + R_2}$$

線形モデル Poly5:
f(x) = p1*x^5 + p2*x^4 + p3*x^3 + p4*x^2 + p5*x + p6
係数 (95% の信頼限界):
p1 = 2.601e+28 (2.297e+28, 2.904e+28)
p2 = -5.616e+23 (-6.164e+23, -5.067e+23)
p3 = 4.683e+18 (4.324e+18, 5.043e+18)
p4 = -1.915e+13 (-2.018e+13, -1.811e+13)
p5 = 4.076e+07 (3.953e+07, 4.199e+07)
p6 = 13.38 (12.93, 13.83)

適合度:
SSE: 9.595
決定係数: 0.9987
自由度調整済み決定係数: 0.9986
RMSE: 0.3195

Curve Fitting Tool

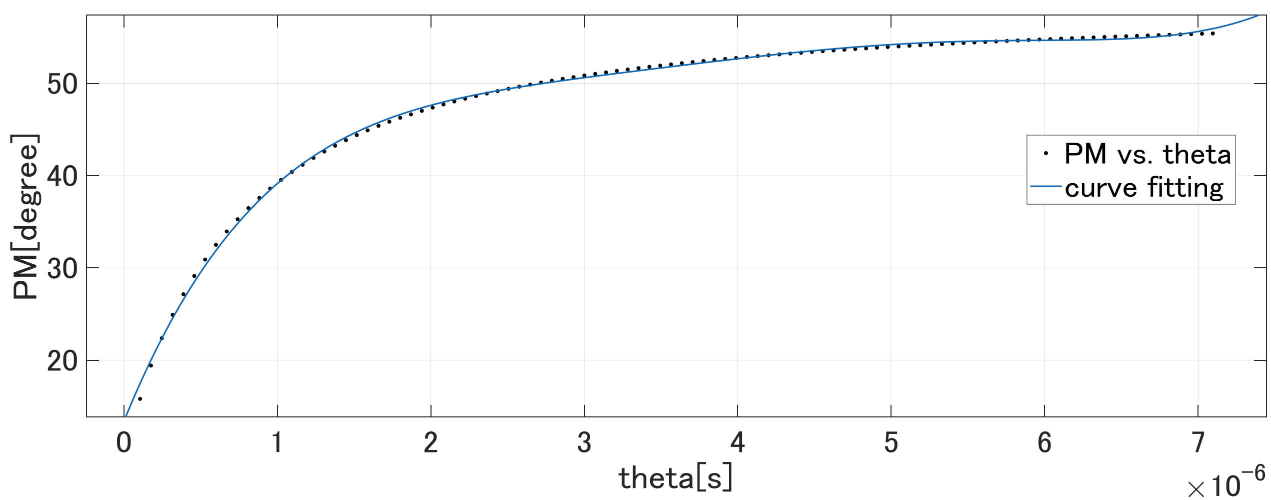
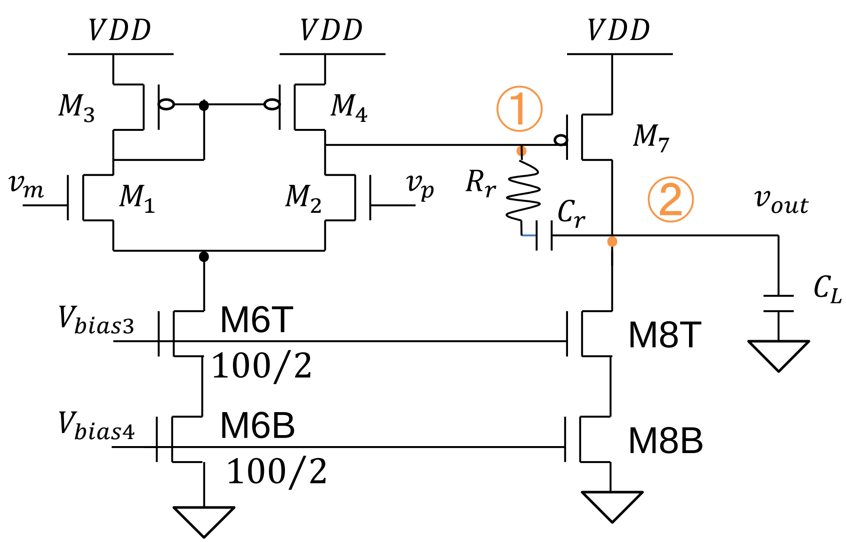
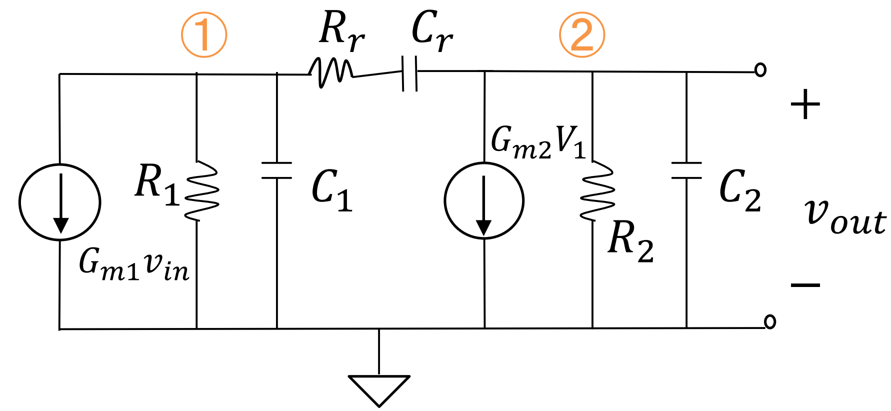


Fig.3 Relationship between PM with parameter θ at feedback factor $f = 0.01$ condition.

Contents



(a) Transistor level circuit



(b) Small-signal model

Fig.4 Two-pole amplifier with compensation of Miller RHP zero

$R_1 = r_{on}||r_{op} = 111k\Omega$

$R_2 = r_{op}||R_{ocasn} \approx r_{op} = 333k\Omega$

$G_{m1} = g_{mn} = 150\,uA/V$

$G_{m2} = g_{mp} = 150\,uA/V$

$C_1 = C_{dg4} + C_{dg2} + C_{gs7} = 13.6fF$

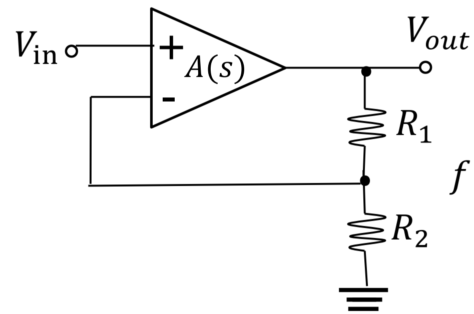
$C_2 = C_L + C_{gd8} \approx C_L + 1.56fF = 101.56fF$

$(C_L = 100fF)$

Routh-Hurwitz Method

Closed-loop transfer function:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{A(s)}{1+fA(s)} = \frac{A_0(1+d_1 s)}{1+fA_0+(a_1+fA_0d_1)s+a_2s^2+a_3s^3}$$



$$f = \frac{R_2}{R_1+R_2}$$

Explicit stability condition of parameters:

$$\alpha = a_1 + fA_0d_1$$

$$= R_1C_1 + R_2C_2 + (R_1+R_2+R_r)C_r + (G_{m2} - fG_{m1} + fG_{m1}G_{m2}R_r)R_1R_2C_r > 0$$

$$\beta = \frac{(a_1+fA_0d_1)a_2 - a_3(1+fA_0)}{a_2} > 0$$

$(parameter\ of\ Routh\ stable)$

α, β: time dimension parameters

Routh table

s^n	α_n	α_{n-2}	α_{n-4}	α_{n-6}	$\cdots$
s^{n-1}	α_{n-1}	α_{n-3}	α_{n-5}	α_{n-7}	$\cdots$
s^{n-2}	$\beta_1 = \frac{\alpha_{n-1}\alpha_{n-2}-\alpha_n\alpha_{n-3}}{\alpha_{n-1}}$	$\beta_2 = \frac{\alpha_{n-1}\alpha_{n-4}-\alpha_n\alpha_{n-5}}{\alpha_{n-1}}$	β_3	β_4	$\cdots$
s^{n-3}	$\gamma_1 = \frac{\beta_1\alpha_{n-3}-\alpha_{n-1}\beta_2}{\beta_1}$	$\gamma_2 = \frac{\beta_1\alpha_{n-5}-\alpha_{n-1}\beta_3}{\beta_1}$	γ_3	γ_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	α_0				

Relationship: α, β and phase margin

Interpolation by MATLAB

Data collection

$$C_{r1}\begin{cases} R_{r11} & (\alpha_{11},\beta_{11}) \\ R_{r12} & (\alpha_{12},\beta_{12}) \\ R_{r13} & (\alpha_{13},\beta_{13}) \\ \cdots & \cdots \\ R_{r19} & (\alpha_{19},\beta_{19}) \end{cases}$$

$$C_{r2}\begin{cases} R_{r21} & (\alpha_{21},\beta_{21}) \\ R_{r22} & (\alpha_{22},\beta_{22}) \\ R_{r23} & (\alpha_{23},\beta_{23}) \\ \cdots & \cdots \\ R_{r29} & (\alpha_{29},\beta_{29}) \end{cases}$$

$$C_{r3}\begin{cases} R_{r31} & (\alpha_{31},\beta_{31}) \\ R_{r32} & (\alpha_{32},\beta_{32}) \\ R_{r33} & (\alpha_{33},\beta_{33}) \\ \cdots & \cdots \\ R_{r39} & (\alpha_{39},\beta_{39}) \end{cases}$$

$$\cdots$$

$$C_{r9}\begin{cases} R_{r91} & (\alpha_{91},\beta_{91}) \\ R_{r92} & (\alpha_{92},\beta_{92}) \\ R_{r93} & (\alpha_{93},\beta_{93}) \\ \cdots & \cdots \\ R_{r99} & (\alpha_{99},\beta_{99}) \end{cases}$$

Produce $9 * 9 = 81$ groups data

Interpolation by MATLAB

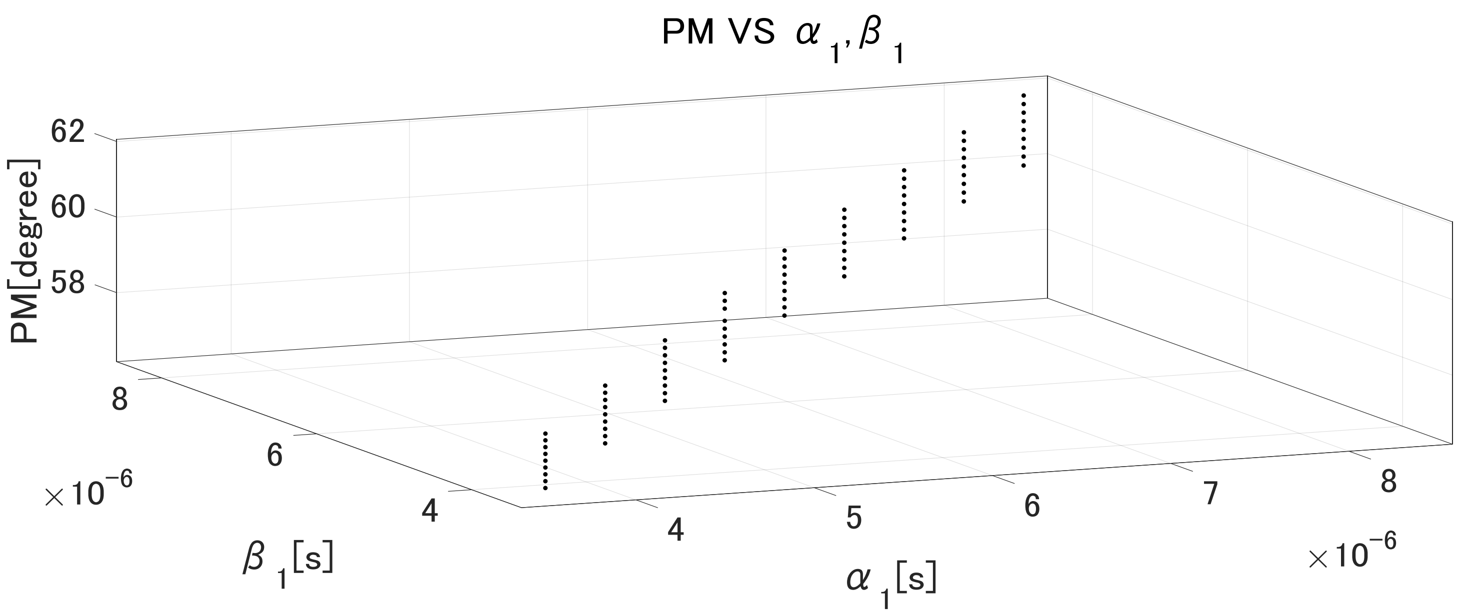
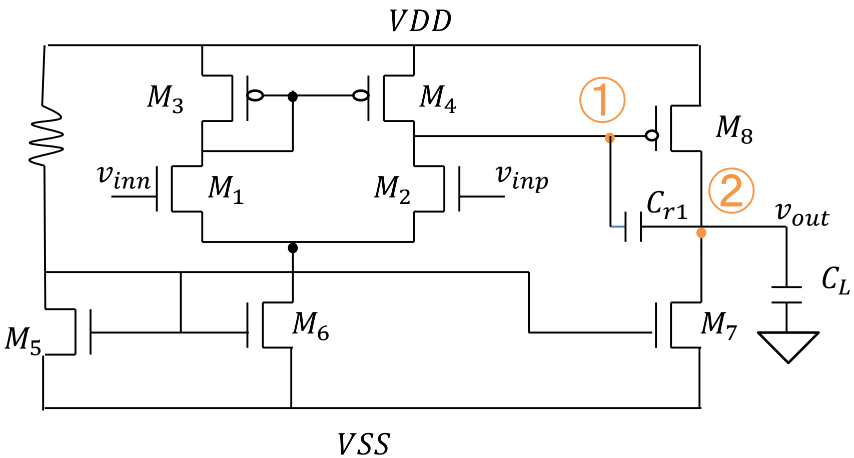


Fig.5 Relationship between PM with parameter α_1, β_1 at feedback factor $f = 0.01$ condition.

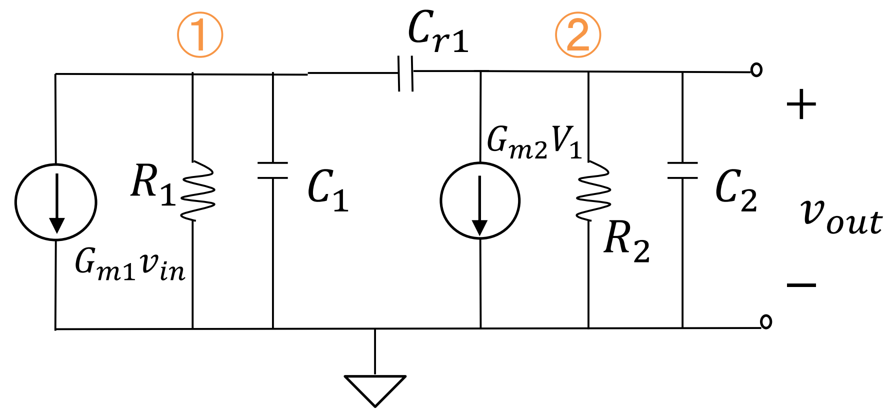
- Monotonic relationship
- Following with the increase of parameter's value
  ⇨ the phase margin will be increased
  feedback system will be more stable

Contents

Verification Circuit

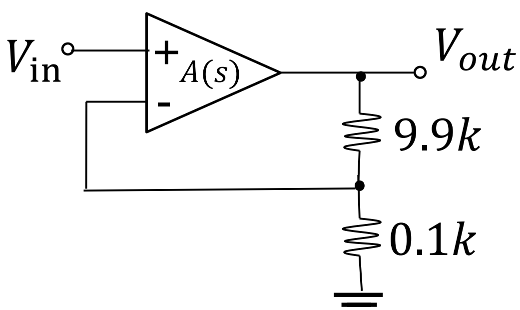


(a) Transistor level circuit



(b) Small-signal model

Fig.6 Two-pole amplifier with inter-stage capacitance



$$f = \frac{0.1}{0.1 + 9.9} = 0.01$$

$$R_1 = r_{on} || r_{op} = 111k\Omega$$

$$R_2 = r_{op} || R_{ocasn} \approx r_{op} = 333k\Omega$$

$$G_{m1} = g_{mn} = 150\, uA/V$$

$$G_{m2} = g_{mp} = 150\, uA/V$$

$$C_1 = C_{dg4} + C_{dg2} + C_{gs7} = 13.6fF$$

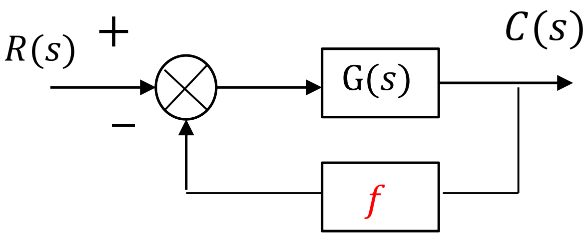
$$C_2 = C_L + C_{gd8} \approx C_L + 1.56fF = 101.56fF$$

$$(C_L = 100fF)$$

Routh-Hurwitz method

Closed-loop transfer function:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{A(s)}{1 + fA(s)} = \frac{A_0(1 + b_1 s)}{1 + fA_0 + (a_1 + fA_0 b_1)s + a_2 s^2}$$



Explicit stability condition of parameters:

$$\theta = a_1 + fA_0 b_1$$
$$= R_1C_1 + R_2C_2 + (R_1 + R_2)C_{r1} + (G_{m2} - fG_{m1})R_1R_2C_{r1} > 0$$

Relationship: C_{r1} and phase margin

MATLAB

Data fitting

Data Fitting by MATLAB

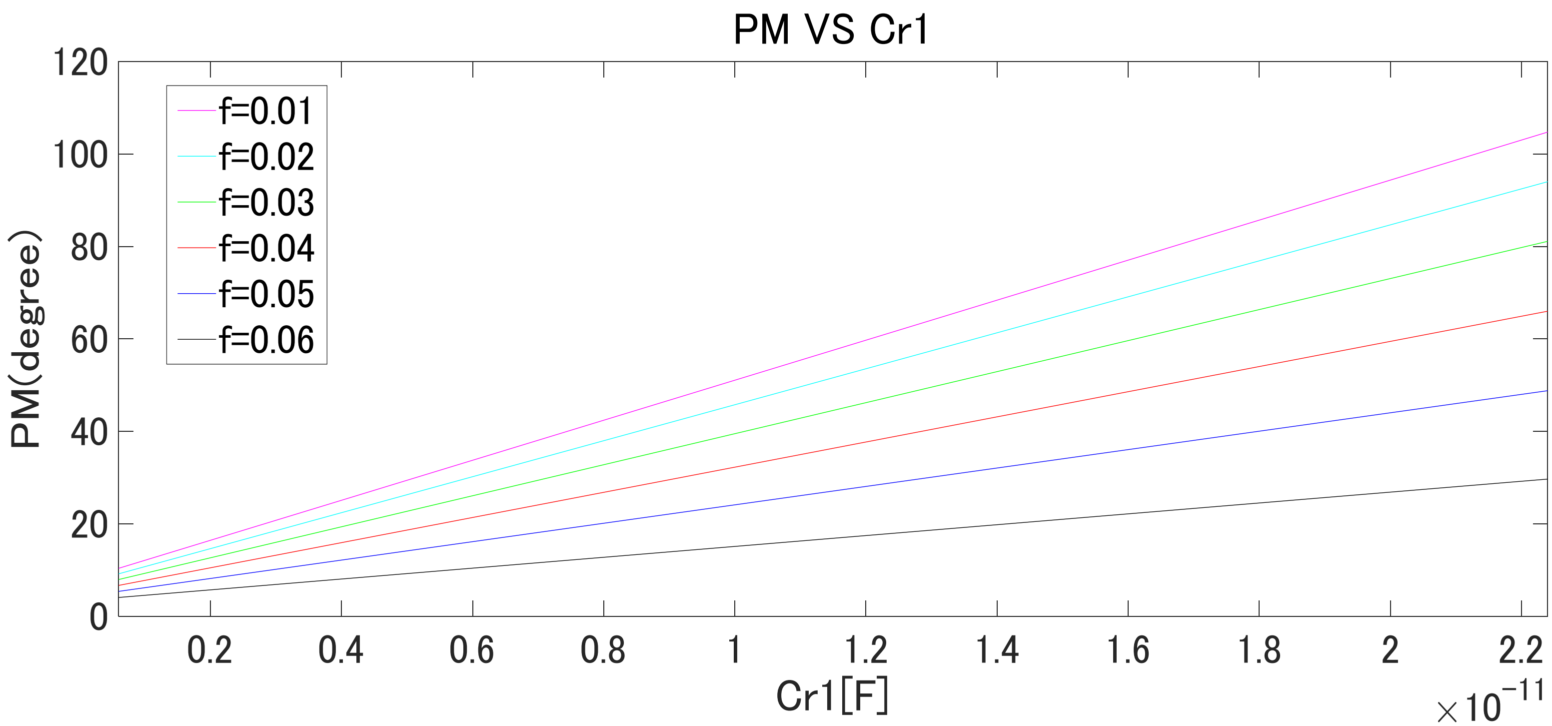
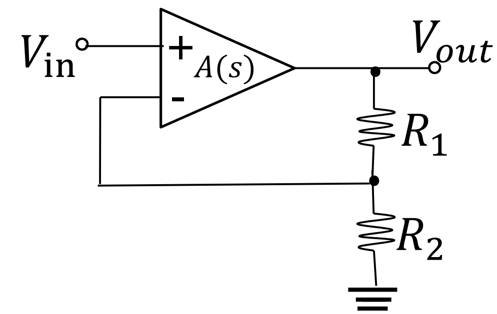


Fig.7 Relationship between PM with compensation capacitor C_{r1} at variation feedback factor f conditions.

PM VS C_{r1}

f = 0.01 condition



$$f = \frac{R_2}{R_1 + R_2} = 0.01$$

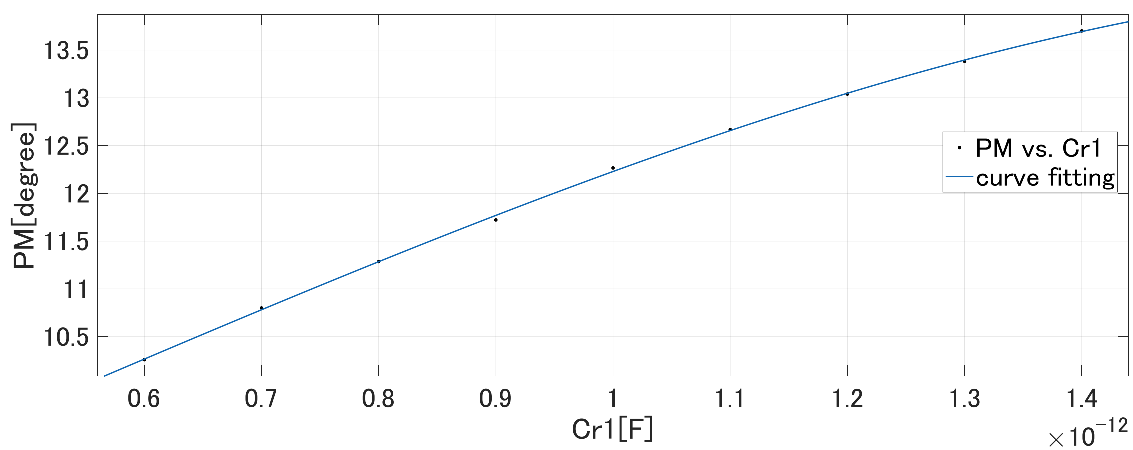
Relation function:

$$\mathrm{PM} = f_1(C_{r1})$$
$$= -1.026e^{36}{C_{r1}}^3 + 1.52e^{24}{C_{r1}}^2 + 4.488e^{12}C_{r1} + 7.247$$

C_{r1}: independent variable
PM: dependent variable

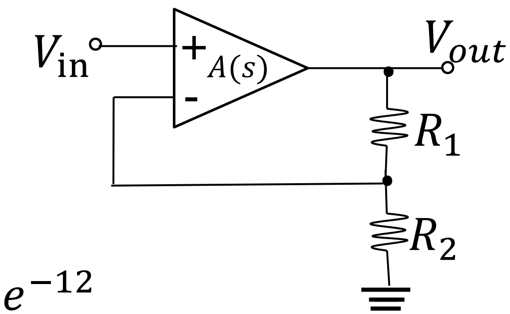
線形モデル Poly3:
  f(x) = p1*x^3 + p2*x^2 + p3*x + p4
係数 (95% の信頼限界):
  p1 = -1.026e+36 (-3.052e+36, 9.994e+35)
  p2 = 1.52e+24 (-4.573e+24, 7.612e+24)
  p3 = 4.488e+12 (-1.415e+12, 1.039e+13)
  p4 = 7.247 (5.412, 9.083)

適合度:
 SSE: 0.004426
 決定係数: 0.9996
 自由度調整済み決定係数: 0.9994
 RMSE: 0.02975



C_{r1} VS PM

f = 0.01 condition



Relation function:

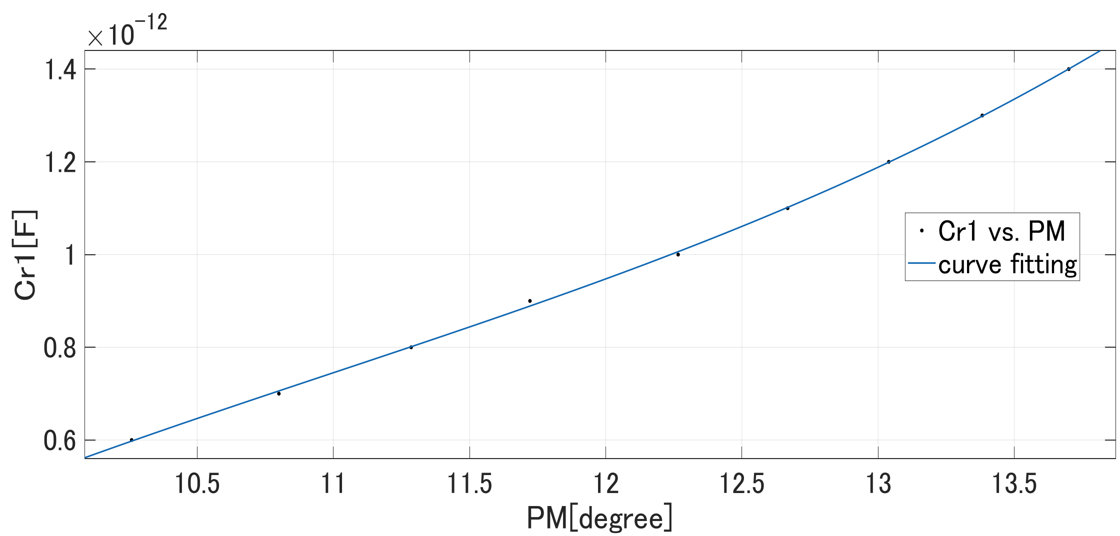
$$C_{r1} = f_1(PM)$$
$$= 6.343e^{-15}PM^3 - 2.091e^{-13}PM^2 + 2.493e^{-12}PM - 9.822e^{-12}$$

C_{r1}: dependent variable
PM: independent variable

$$f = \frac{R_2}{R_1 + R_2} = 0.01$$

線形モデル Poly3:
f(x) = p1*x^3 + p2*x^2 + p3*x + p4
係数 (95% の信頼限界):
p1 = 6.343e-15 (8.692e-16, 1.182e-14)
p2 = -2.091e-13 (-4.059e-13, -1.223e-14)
p3 = 2.493e-12 (1.435e-13, 4.843e-12)
p4 = -9.822e-12 (-1.913e-11, -5.162e-13)

適合度:
SSE: 2.085e-28
決定係数: 0.9997
自由度調整済み決定係数: 0.9994
RMSE: 6.457e-15



For stable feedback system,
necessary PM value: 45 degree or 60 degree

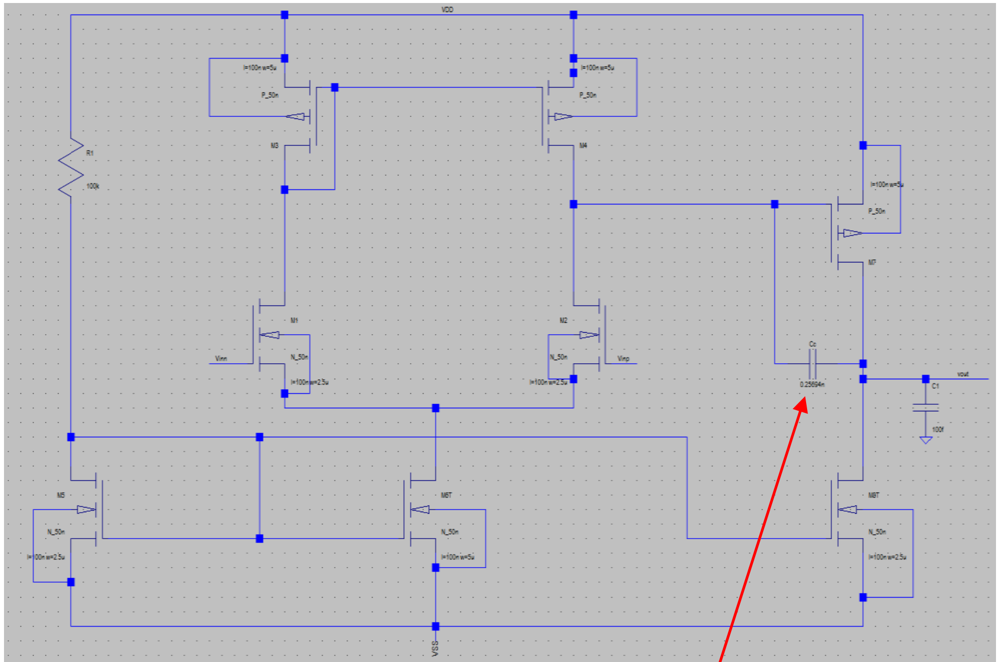
$$C_{r1} = f_1(PM)$$
$$= 6.343e^{-15}PM^3 - 2.091e^{-13}PM^2 + 2.493e^{-12}PM - 9.822e^{-12}$$

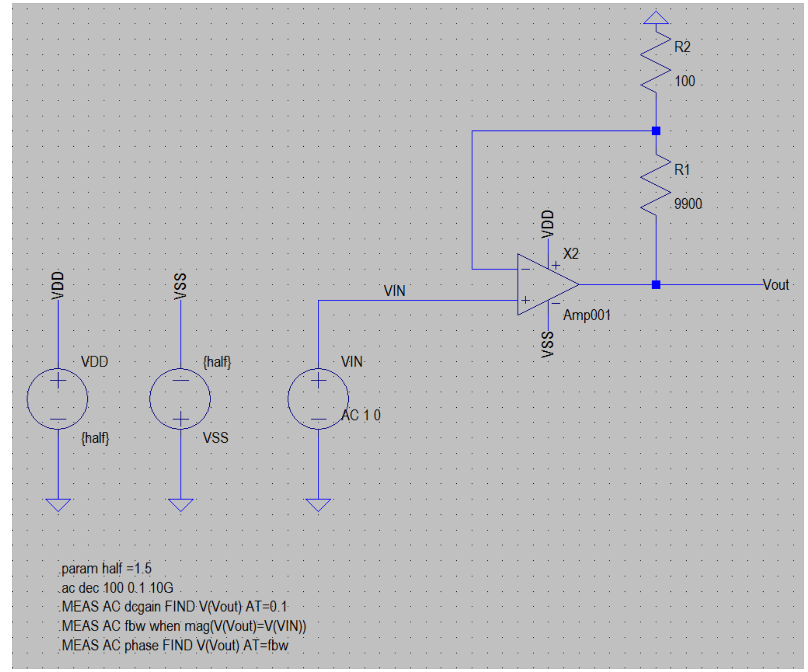
PM=45degree, $C_{r1} = 2.5694e^{-10}F = 0.25694\text{nF}$

PM=60degree, $C_{r1} = 7.5709e^{-10}F = 0.75709\text{nF}$

For stability and needed PM value,
compensation capacitance can be calculated.

Simulation by LTspice





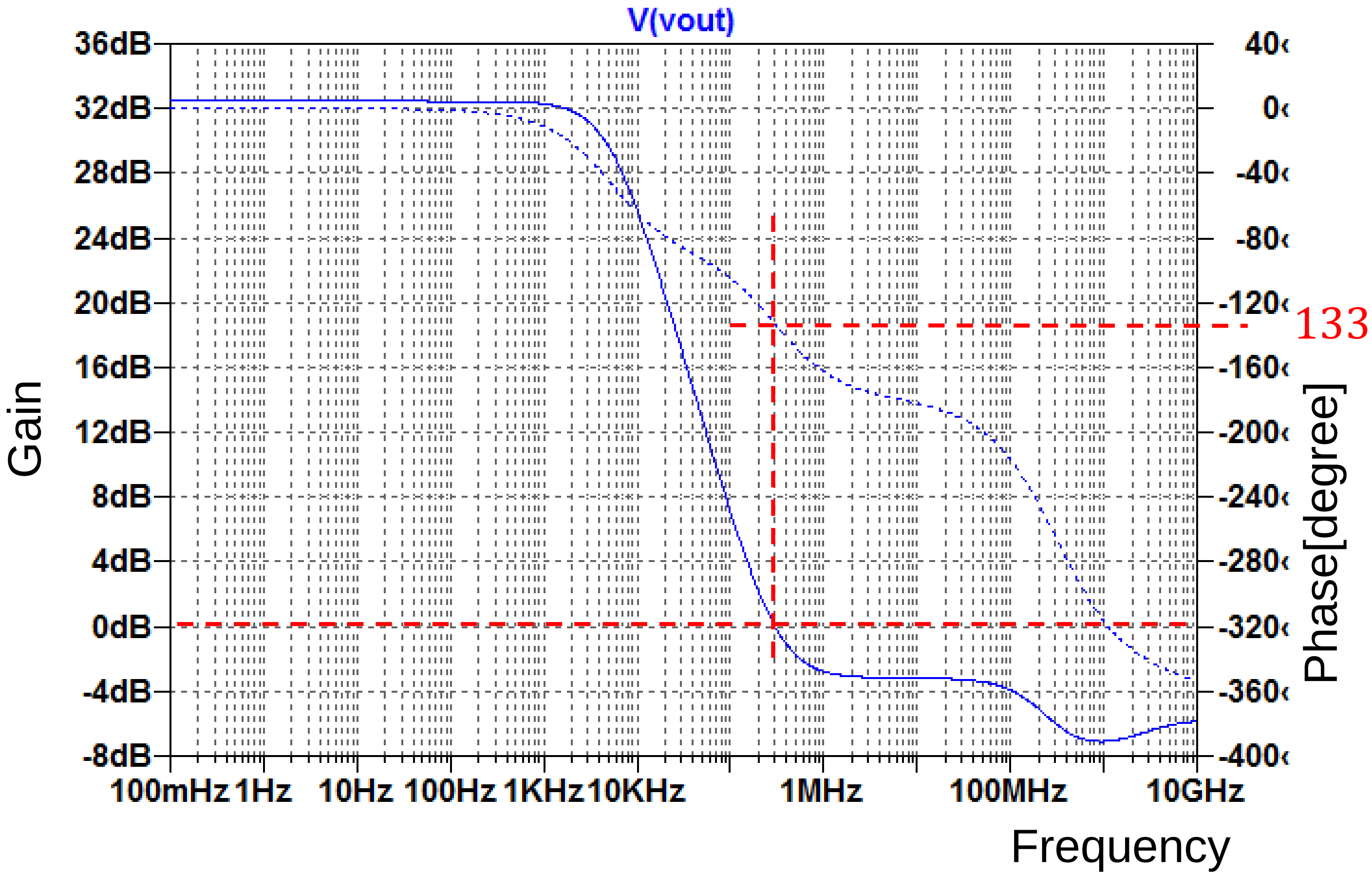
compensation capacitor:

$$C_{r1} = 0.25694nF$$

feedback factor:

$$f = \frac{0.1k}{9.9k} = 0.01$$

Simulation result




```
phase: v(vout)=(-4.66844e-005dB,-133.013°) at 301437
```

$$Phase\ Margin = 180^{\circ} - 133^{\circ} = 47^{\circ}$$

Contents

- Research Objective & Background
- Stability Criteria
  - Nyquist Criterion
  - Routh-Hurwitz Criterion
- Relationship between Routh-Hurwitz criterion parameter with phase margin
- Simulation Verification
- Discussion & Conclusion

Discussion

Depict small signal equivalent circuit of amplifier

↓

Derive open-loop transfer function

↓

Derive closed-loop transfer function
& obtain characteristics equation

↓

Apply R-H stability criterion
& obtain explicit stability condition

Especially effective for

Multi-stage opamp (high-order system)

Limitation

Explicit transfer function with polynomials of s has to be derived.

Conclusion

- R-H method, explicit circuit parameter conditions can be obtained for feedback stability.
- Relationship between R-H criterion parameter with phase margin:
  (1) monotonic relationship
  (2) the system will be more stable, following with the increase of parameter's value.
- The proposed method has been confirmed with LTspice simulation

↓

R-H method can be used
with conventional Bode plot method.

Thank you

for your kind attention.

Q&A

Q: Bode plot often be used for judging stability, you propose use Routh-Hurwitz method. what' the difference between two methods.

A: Traditional method, Bode plot often be used, it can show stability information directly. From Bode plot, we can only find out the stability, judge the system stable or unstable. But we don't know internal connection between circuit parameters and stability. we don't know which parameter values influence the stability, capacitor, resistor.
Used the proposed method , we can obtain explicit stability condition for circuit parameters, and we can find out the that between R-H criterion parameter with phase margin is Monotonic relationship.

At our previous research, we have verified the equivalence that between R-H criterion with Nyquist criterion