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# Design of Operational Amplifier Phase Margin Using Routh-Hurwitz Method

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**Abstract.** This paper describes a method that uses Routh-Hurwitz stability criterion for analysis and design of the operational amplifier stability and phase margin. Theoretically, the relationship between Routh-Hurwitz criterion parameters with phase margin of the operational amplifier is deduced, also the proposed method is verified with SPICE simulations of two operational amplifiers at transistor level circuits.

## 1. Introduction

We proposed to use the Routh-Hurwitz (R-H) stability criterion for operational amplifier stability analysis and design, to obtain explicit stability conditions for operational amplifier circuit parameters [1]; this has not been described in any operational design book/paper, to the best of our knowledge [3-5]. Also we showed the equivalency of the Nyquist and Routh-Hurwitz stability criteria [2]. In this paper, we explore the relationships between Routh-Hurwitz criterion parameters with the phase margin of the operational amplifier as theoretical support and perfection for the proposed method. At the verification part, our SPICE simulation results show its good agreements with our theoretical analysis based on the proposed method.

## 2. Routh-Hurwitz stability criterion

Suppose that characteristic equation of the closed-loop transfer function is as follows:

$$\boldsymbol{D(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0} \qquad (1)$$

Table 1. Routh table.

| | | | | | |
|---|---|---|---|---|---|
| $s^n$ | $\alpha_n$ | $\alpha_{n-2}$ | $\alpha_{n-4}$ | $\alpha_{n-6}$ | $\cdots$ |
| $s^{n-1}$ | $\alpha_{n-1}$ | $\alpha_{n-3}$ | $\alpha_{n-5}$ | $\alpha_{n-7}$ | $\cdots$ |
| $s^{n-2}$ | $\beta_1 = \frac{\alpha_{n-1}\alpha_{n-2} - \alpha_n\alpha_{n-3}}{\alpha_{n-1}}$ | $\beta_2 = \frac{\alpha_{n-1}\alpha_{n-4} - \alpha_n\alpha_{n-5}}{\alpha_{n-1}}$ | $\beta_3$ | $\beta_4$ | $\cdots$ |
| $s^{n-3}$ | $\gamma_1 = \frac{\beta_1\alpha_{n-3} - \alpha_{n-1}\beta_2}{\beta_1}$ | $\gamma_2 = \frac{\beta_1\alpha_{n-5} - \alpha_{n-1}\beta_3}{\beta_1}$ | $\gamma_3$ | $\gamma_4$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $s^0$ | $\alpha_0$ | | | | |

Necessary and sufficient condition of the stability is that all real parts of the solutions of (1) are negative, which is equivalent as follows:

$\boldsymbol{\alpha_i > 0}$ for i=0, 1, ... , n, and all values of the first column parameters in Routh table (Table 1) are positive.

## 3. Relationship between R-H parameters and phase margin

**Amplifier1:**

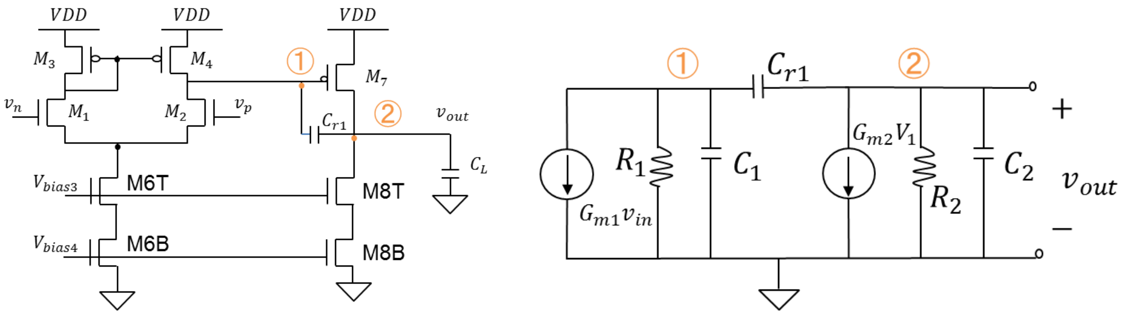



Fig. 1 Two-stage amplifier with inter-stage capacitance

$\boldsymbol{R_1, R_2}$ are equivalent resistors, $\boldsymbol{C_1, C_2}$ are equivalent capacitances, $\boldsymbol{G_{m1}, G_{m2}}$ are transconductances, $\boldsymbol{C_{r1}}$ is compensation capacitance, $\boldsymbol{v_{in} = v_p - v_n}$.

Consider the two-stage amplifier in Fig. 1 whose open-loop transfer function is given by

$$\boldsymbol{G(s) = \frac{K(1+b_1 s)}{1+a_1 s+a_2 s^2}} \tag{2}$$

and the parameters are provided by:

$$\boldsymbol{b_1 = -\frac{C_r}{G_{m2}},\ K = G_{m1}G_{m2}R_1R_2,\ a_1 = R_1C_1 + R_2C_2 + (R_1 + R_2 + R_1G_{m2}R_2)C_{r1},}$$
$$\boldsymbol{a_2 = R_1R_2C_2\left[C_1 + \left(1 + \frac{C_1}{C_2}\right)C_{r1}\right]} \tag{3}$$

Accordingly, Fig. 2 shows a feedback amplifier using this operational amplifier using sufficiently large R1, R2, and its closed-loop transfer function is given as follows:

$$\boldsymbol{H(s) = \frac{G(s)}{1+fG(s)} = \frac{K+Kb_1 s}{1+fK+(a_1+fKb_1)s+a_2 s^2}} \tag{4}$$

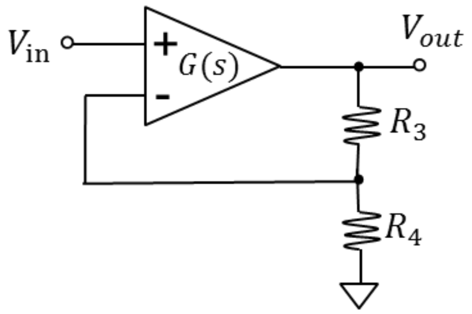



Fig. 2 Feedback system with $\boldsymbol{f = R_4/(R_3 + R_4)}$.

Based on the R-H stability criterion, we can obtain the following explicit condition as the necessary and sufficient condition for the operational amplifier feedback circuit stability:

$$\boldsymbol{\theta = a_1 + fKb_1 = R_1C_1 + R_2C_2 + (R_1 + R_2)C_{r1} + (G_{m2} - fG_{m1})R_1R_2C_{r1} > 0} \tag{5}$$

We define the R-H parameter $\boldsymbol{\theta}$ as one dimension parameter. Using the parameter values of short-channel CMOS devices ( $\boldsymbol{R_1 = 111\mathrm{k}\Omega}$, $\boldsymbol{R_2 = 333\mathrm{k}\Omega}$, $\boldsymbol{C_1 = 13.6fF}$, $\boldsymbol{C_2 = 101.56fF}$, $\boldsymbol{G_{m1} = 100\,uA/V}$, $\boldsymbol{G_{m2} = 180\,uA/V}$ ), and assigning the compensation capacitance $\boldsymbol{C_{r1}}$, calculating the corresponding operational amplifier system phase margin (PM), gain margin (GM), $\boldsymbol{F_{gm}}$ and $\boldsymbol{F_{pm}}$ at various feedback factor $\boldsymbol{f}$ conditions, using equation (2) by MATLAB, and then calculating the corresponding values of parameter $\boldsymbol{\theta}$ using equation (5). $\boldsymbol{F_{gm}}$ is the frequency where the gain margin is obtained with MATLAB simulation, and $\boldsymbol{F_{pm}}$ is the frequency where the phase margin is obtained. We can obtain the values as Table 2 (at condition $\boldsymbol{f = 0.01}$ ).

Table 2. Data collection.

| $f$=0.01 | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Cr [fF] | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | … |
| θ [uS] | 0.11 | 0.18 | 0.25 | 0.32 | 0.39 | 0.46 | 0.53 | 0.60 | 0.67 | … |
| PM [degree] | 16 | 19 | 22 | 24 | 27 | 29 | 31 | 33 | 34 | … |
| GM [dB] | 9.1 | 7.6 | 7.0 | 6.6 | 6.4 | 6.3 | 6.2 | 6.0 | 6.0 | … |
| $F_{gm}$ [GHz] | 4.5 | 3.4 | 2.9 | 2.6 | 2.3 | 2.1 | 2.0 | 1.9 | 1.8 | … |
| $F_{pm}$ [GHz] | 2.6 | 2.1 | 1.8 | 1.5 | 1.4 | 1.2 | 1.1 | 1.0 | 9.4 | … |

Using the polyfit function of MATLAB, we can obtain the fitted curve which can indicate the relationship between parameter $\boldsymbol{\theta}$ with phase margin as shown in Fig.3 at various feedback factor conditions. At the feedback factor $\boldsymbol{f = 0.01}$ condition, we can obtain the fitted curve as Fig.4, and corresponding relation function is given as follows:

$$\boldsymbol{PM = 2.601e^{28}\theta^5 - 5.616e^{23}\theta^4 + 4.683e^{18}\theta^3 - 1.915e^{13}\theta^2 + 4.076e^{28}\theta + 13.38} \quad (6)$$

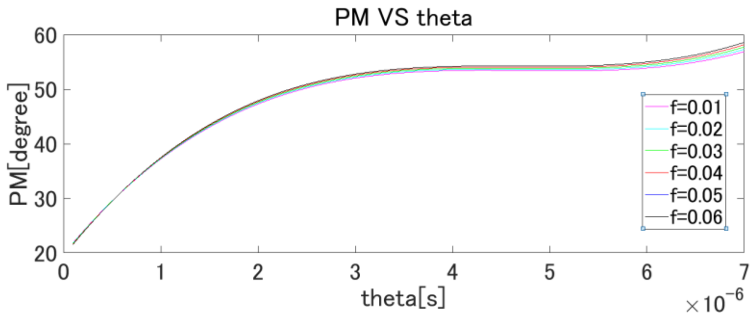



Fig.3 Relationship between PM and parameter $\boldsymbol{\theta}$ at various feedback factor conditions.

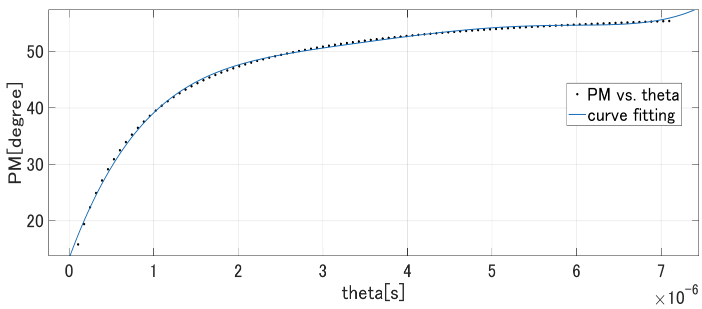



Fig.4 Relationship between PM and parameter $\boldsymbol{\theta}$ at feedback factor $\boldsymbol{f = 0.01}$ condition.

As shown in Fig.3 and Fig.4, the PM and the R-H parameter $\boldsymbol{\theta}$ have the one-to-one relationship. Following with increase of the parameter value, the phase margin will be increased; in other words, the feedback system becomes more stable.

We can calculate a required value of the compensation capacitor, for a given operational amplifier PM, based on the calculated value of the parameter $\boldsymbol{\theta}$.

**Amplifier2:**

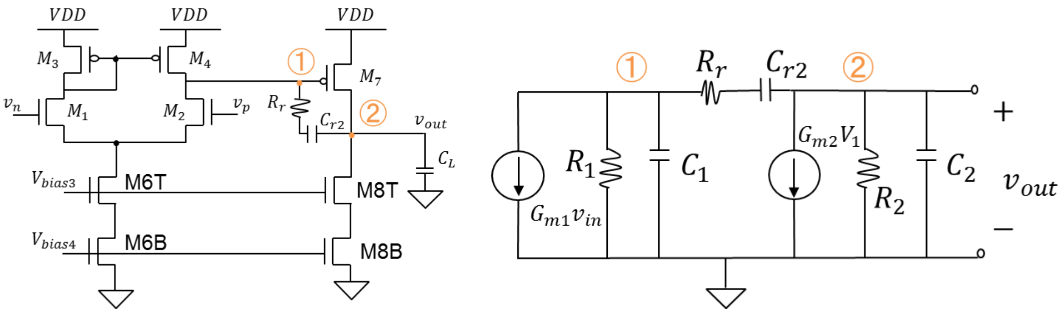



Fig. 5 Two-stage amplifier with compensation network using a nulling resistor.

$\boldsymbol{R_1, R_2}$ are equivalent resistors, $\boldsymbol{C_1, C_2}$ are equivalent capacitances, $\boldsymbol{G_{m1}, G_{m2}}$ are transconductances, $\boldsymbol{C_{r1}}$is compensation capacitance, $\boldsymbol{R_r}$ is compensation resistor, $\boldsymbol{v_{in} = v_p - v_n}$.

Consider the two-stage amplifier in Fig. 5 whose open-loop transfer function is given by

$$\boldsymbol{G(s) = \frac{K(1+b_1 s)}{1+a_1 s+a_2 s^2+a_3 s^3}} \tag{7}$$

and the parameters are:

$$\begin{gathered}\boldsymbol{b_1 = -\left(\frac{C_{r2}}{G_{m2}} - R_r C_{r2}\right),\ K = G_{m1}G_{m2}R_1R_2,}\\ \boldsymbol{a_1 = \ R_1C_1 + R_2C_2 + (R_1 + R_2 + R_r\ +\ R_1R_2G_{m2})C_{r2},}\\ \boldsymbol{a_2 = R_1R_2(C_2C_{r2} + C_1C_2 + C_1C_{r2}) + R_rC_{r2}(R_1C_1 + R_2C_2),\ a_3 = R_1R_2R_rC_1C_2C_{r2}.}\end{gathered} \tag{8}$$

Accordingly, Fig. 1 shows a feedback amplifier using this operational amplifier, and its closed-loop transfer function is given by

$$\boldsymbol{H(s) = \frac{G(s)}{1+fG(s)} = \frac{K+Kb_1 s}{1+fK+(a_1+fKb_1)s+a_2s^2+a_3s^3}} \tag{9}$$

Based on the R-H stability criterion, we can obtain the following explicit condition as the necessary and sufficient condition for the operational amplifier feedback circuit stability:

$$\boldsymbol{\alpha = a_1 + fKd_1 > 0} \tag{10}$$

$$\boldsymbol{\beta = \frac{a_2(a_1+fKb_1\ )-a_3(1+fK\ )}{a_2} > 0} \tag{11}$$

Finally, we can obtain:

$$\boldsymbol{\alpha = R_1C_1 + R_2C_2 + (R_1 + R_2 + R_r)C_{r2} + (G_{m2} - fG_{m1} + fG_{m1}G_{m2}R_r)R_1R_2C_{r2}} \tag{12}$$

$$\boldsymbol{\beta} = \boldsymbol{\alpha} - \frac{R_1R_2C_1C_2R_rC_{r2}(1+fG_{m1}G_{m2}R_1R_2)}{R_1R_2(C_2C_{r2}+C_1C_2+C_1C_{r2})+R_rC_{r2}(R_1C_1+R_2C_2)}$$

(13)

We also define the R-H parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ as dimension parameters. We have calculated the values of parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and PM. At feedback factor $\boldsymbol{f = 0.01}$ condition, we can obtain the relation function in Fig.6, by using interpolation function in the curve fitting tool of MATLAB.

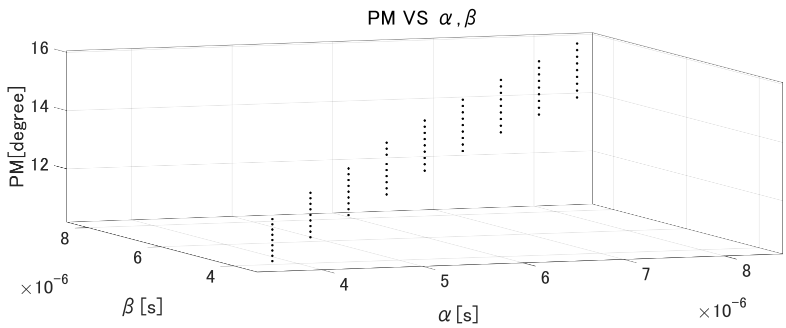



Fig.6 Relationship between PM with parameter $\boldsymbol{\alpha}, \boldsymbol{\beta}$ at feedback factor $\boldsymbol{f = 0.01}$ condition.

As shown in Fig.6, when the value of parameter $\boldsymbol{\alpha}$ is fixed, the PM is increased following with the growing of parameter $\boldsymbol{\beta}$, and similarly when the value of parameter $\boldsymbol{\beta}$ is fixed, the PM is also increased following with the growing of parameter $\boldsymbol{\alpha}$. So the relationship between R-H parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and PM is linear one, and following with the increase of parameter value, the phase margin will be increased; in other words, the feedback system becomes more stable.

## 4. Simulation Verification

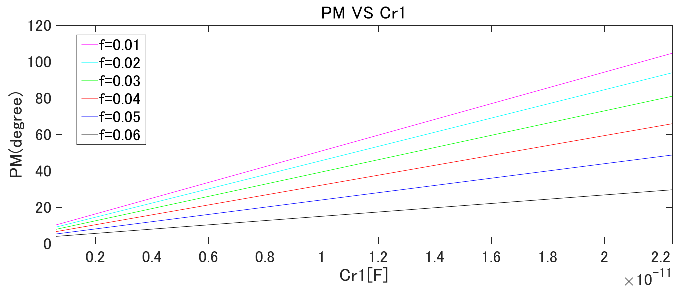
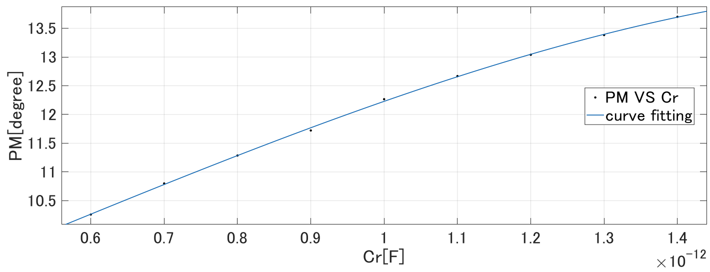
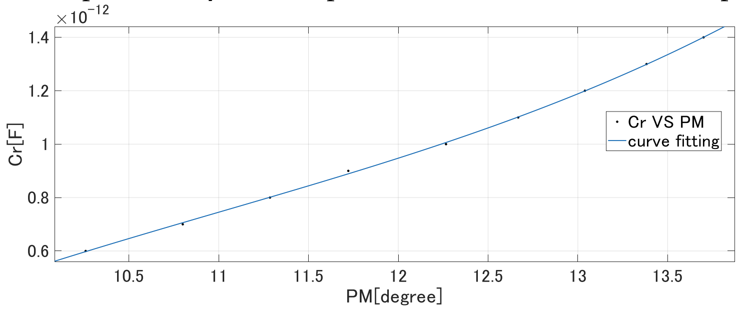



Fig.8 Relationship between PM with compensation capacitor $\boldsymbol{C_r}$ at various feedback factor conditions.

Consider the two-stage amplifier in Fig. 1. Based on the principle and processing represented in Section 2, we can obtain the parameter $\boldsymbol{\theta}$ as shown in (5). We can calculate a required value of the compensation capacitor, for a given operational amplifier PM, based on the calculated value of the parameter $\boldsymbol{\theta}$. Using the polyfit function, we can obtain the curves which can indicate the relationships between capacitor $\boldsymbol{C_r}$ and phase margin in Fig.8.




(a) Compensation capacitor $\boldsymbol{C_r}$ as independent variable and PM as dependent variable.




(b) PM as independent variable and compensation capacitor $\boldsymbol{C_r}$ as dependent variable.
Fig.9 Relationship between PM with compensation capacitor $\boldsymbol{C_r}$ at feedback factor $\boldsymbol{f = 0.01}$ condition.

At the feedback factor $\boldsymbol{f = 0.01}$ condition, we can obtain the fitted curve as shown in Fig.9. By calculation, we can obtain the relation function between PM with capacitor as follows:

$$\boldsymbol{PM = -1.026e^{36}{C_r}^3 + 1.52e^{24}{C_r}^2 + 4.488e^{12}C_r + 7.24} \tag{14}$$

$$\boldsymbol{Cr = 6.343e^{-15}PM^3 - 2.091e^{-13}PM^2 + 2.493e^{-12}PM - 9.822e^{-12}} \tag{15}$$

If we need to obtain $\boldsymbol{45^o}$ phase margin, the required corresponding capacitor value is $\boldsymbol{0.25694\mathrm{nF}}$ by calculation based on Eq. (15).

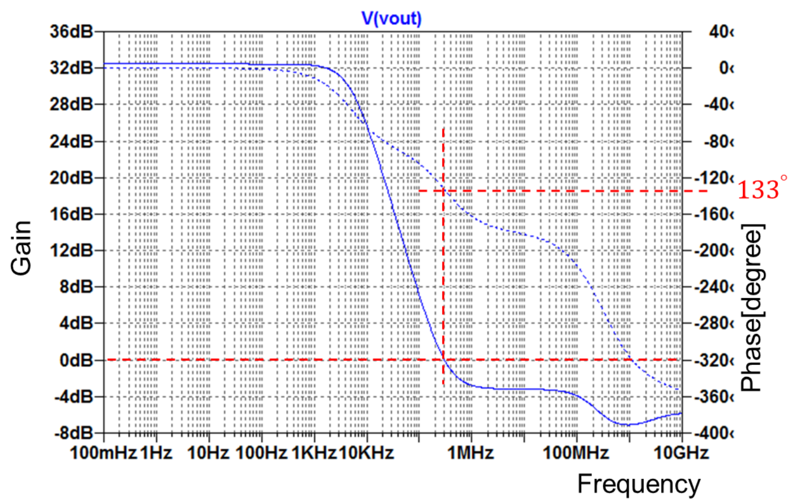



Fig.10 LTspice simulation result at conditions: feedback factor $\boldsymbol{f = 0.01}$, compensation capacitance of $\boldsymbol{0.25694\mathrm{nF}}$.

For verify this result, we make simulation used amplifier circuit shown in Fig.1, the feedback system circuit shown in Fig.2 for the feedback factor $\boldsymbol{f = 0.01}$, and the compensation capacitance value is

$\mathbf{0.25694nF}$. The simulation result is shown in Fig.8. The phase margin result is $\mathbf{180^{\circ} - 133^{\circ} = 47^{\circ}}$ obtained from LTspice simulation, which is close to the calculation result of $\mathbf{45^{\circ}}$ based on Eq. (15).

The relationship between Cr1 and the phase margin (corresponding to Fig.8) can be obtained by using the small equivalent circuit, which can indicate the variation tendency of stability following circuit parameter's variation. However, as we see, this relationship only can reflect the impact from a single circuit parameter on stability. The advantages of the proposed method are the explicit stability conditions (5), (12), (13) and the relationship between parameters and phase margin (corresponding function (6) and Fig.4), we can overall consider multiple circuit parameters one time as well as the trade-off analysis between the influences on system stability from every single circuit parameter.

## 5. Discussion

It may be true that derivation of precise explicit transfer function with polynomials of $\boldsymbol{s}$ is difficult due to many parasitic components in the operational amplifier circuit. However, even if the derived equivalent circuit or transfer function uses only major components and neglects parasitic components, the R-H method provides the information about which major parameter values should be increased or decreased for better stability and larger phase margin. Hence the usage of R-H method together with the conventional Bode plot would be useful for the operational amplifier phase margin design.

As some delays exist in the system, so there are small deviations between the experimental and calculation results.

## 6. Conclusion

This paper has deduced the relationship between Routh-Hurwitz criterion parameters with phase margin of the operational amplifier, and has shown that they are monotonic relationship. In the verification part, satisfactory results have been obtained with LTspice simulations at transistor level circuit.

The R-H method has an advantage of being able to obtain explicit stability condition for circuit parameters; hence we expect that the R-H method can be practically used together with the Bode plot method.

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