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Design of Operational Amplifier Phase Margin Using Routh-Hurwitz Method

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Contents

- Research Objective & Background
- Stability Criteria
  - Nyquist Criterion
  - Routh-Hurwitz Criterion
- Equivalence at Mathematical Foundations
- Relationship between R-H parameters and phase margin
- Simulation Verification
- Discussion & Conclusion

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Research Background (Stability Theory)

- **Electronic Circuit Design Field**
  - Bode plot (>90% frequently used)
  - Nyquist plot

- **Control Theory Field**
  - Bode plot
  - Nyquist plot
  - Nicholas plot
  - Routh-Hurwitz stability criterion

    ⇨ Very popular in control theory field
    but rarely seen in electronic circuit books/papers
  - Lyapunov function method

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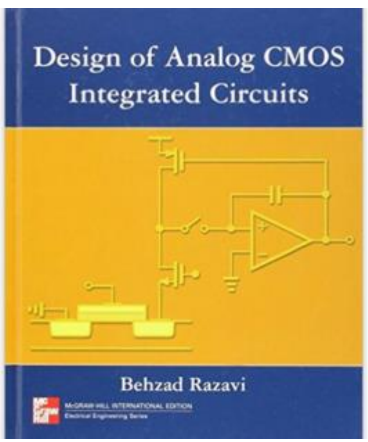
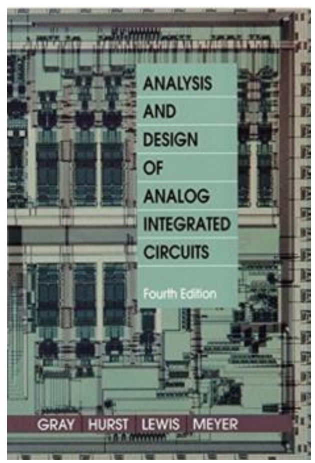
Electronic Circuit Text Book

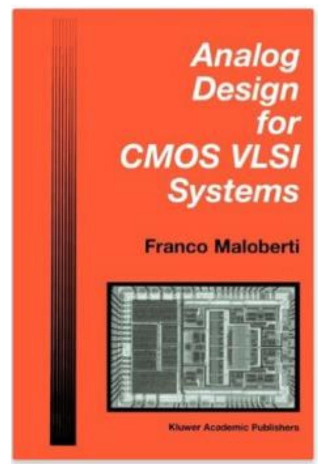
We were NOT able to find out any electronic circuit text book which describes Routh-Hurwitz method
for operational amplifier stability analysis and design !



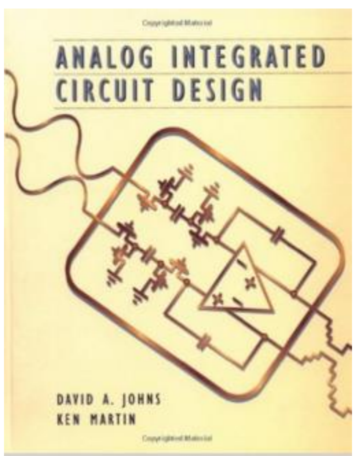
Razavi



Gray



Maloberti

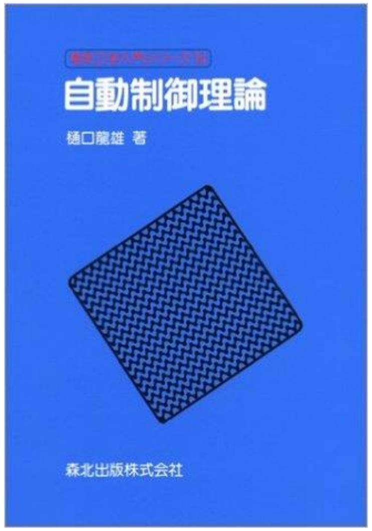


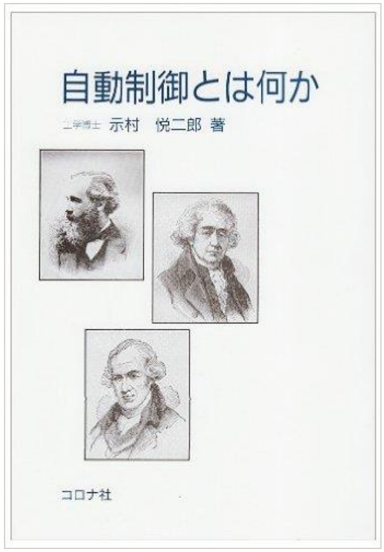
Martin

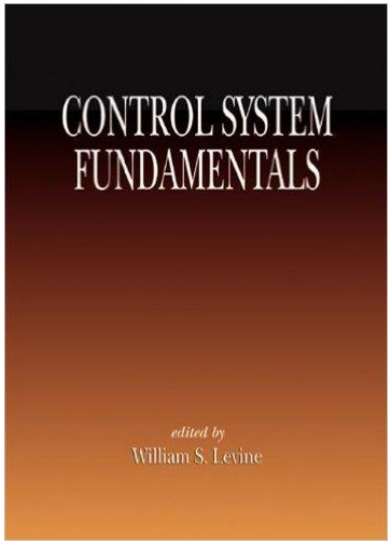
None of the above describes Routh-Hurwitz.
Only Bode plot is used.

Control Theory Text Book

Most of control theory text books
describe Routh-Hurwitz method
for system stability analysis and design !







Research Objective

Our proposal

For

Analysis and design of operational amplifier stability and phase margin

Use

Routh-Hurwitz stability criterion

We can obtain

- Explicit stability condition for circuit parameters (which can NOT be obtained only with Bode plot).
- Relationship between R-H parameters and phase margin

Contents

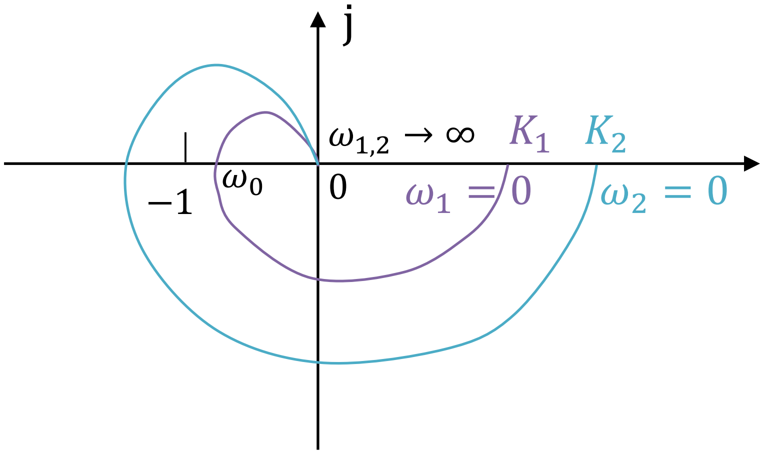
Nyquist plot

- Open-loop frequency characteristic

  ➡ Closed-loop stability

- Necessary and sufficient condition :

  When $\omega = 0 \rightarrow \infty$, $\quad N = P - Z$



Nyquist plot of open-loop system

N : number, Nyquist plot anti-clockwise encircle point (-1,j0).

P: number, positive roots of open-loop characteristic equation.

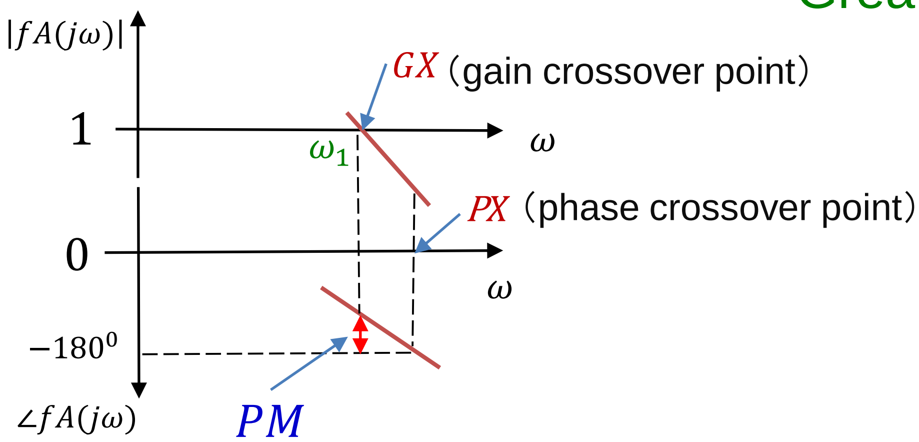
Z: number, positive roots of closed-loop characteristic equation.

- If the open-loop system is stable(P=0), the Nyquist plot mustn't encircle the point (-1,j0).

$$\angle G_{open}(j\omega_0) = -\pi, \left|G_{open}(j\omega_0)\right| < 1$$

Bode Plot

GX precedes PX ⇨ Feedback system is stable



Greater spacing between GX and PX

⇩

More stable

ω_1: gain crossover frequency

Phase margin : $\mathrm{PM} = 180^0 + \angle fA(\omega = \omega_1)$

Bode plot is useful,
but it does NOT show explicit stability conditions of circuit parameters.

Phase Margin and Gain Margin

φ: Phase Margin

- The included angle
  - Negative real axis
  - Intersection point at ω_c

- The angle difference
  - Phase at ω_c
  - -180^0

h: *Gain Margin*

- Reciprocal $\dfrac{1}{|G(\omega_g)|}$

- The distance
  - Gian at ω_g
  - 0dB, real axis

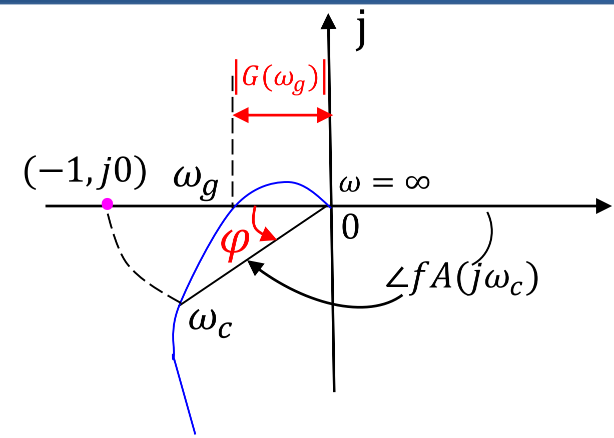


Fig.(a) Nyquist plot

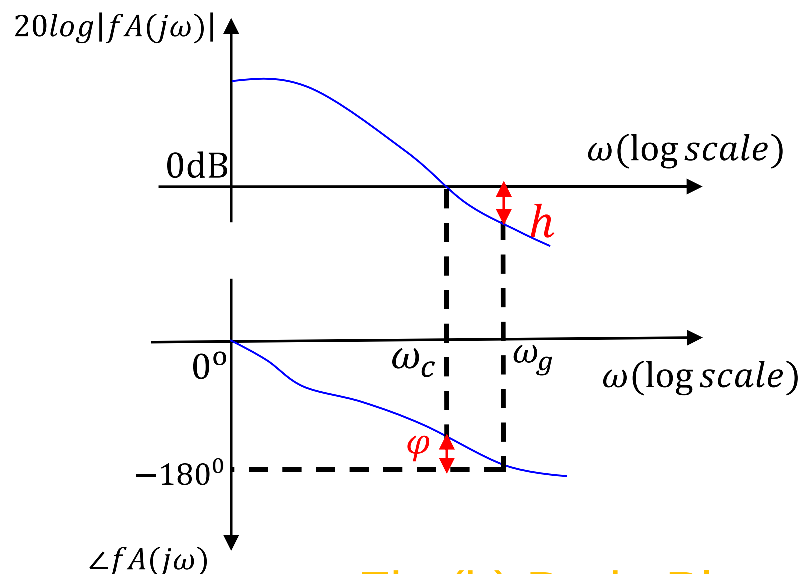


Fig.(b) Bode Plot

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Routh Stability Criterion

Characteristic equation:

$$D(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0$$

Sufficient and necessary condition:

(i) $\alpha_i > 0 \;\text{ for } i = 0,1,\ldots,n$

&

(ii) All values of Routh table's first columns are positive.

Routh table

s^n	α_n	α_{n-2}	α_{n-4}	α_{n-6}	$\cdots$
s^{n-1}	α_{n-1}	α_{n-3}	α_{n-5}	α_{n-7}	$\cdots$
s^{n-2}	$\beta_1 = \frac{\alpha_{n-1}\alpha_{n-2} - \alpha_n\alpha_{n-3}}{\alpha_{n-1}}$	$\beta_2 = \frac{\alpha_{n-1}\alpha_{n-4} - \alpha_n\alpha_{n-5}}{\alpha_{n-1}}$	β_3	β_4	$\cdots$
s^{n-3}	$\gamma_1 = \frac{\beta_1\alpha_{n-3} - \alpha_{n-1}\beta_2}{\beta_1}$	$\gamma_2 = \frac{\beta_1\alpha_{n-5} - \alpha_{n-1}\beta_3}{\beta_1}$	γ_3	γ_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	α_0				

Mathematical test

Determine whether given polynomial has all roots in the left-half plane.

Contents

Four Examples

Ex.1 $G(s) = \dfrac{K}{1 + a_1 s + a_2 s^2 + a_3 s^3}$ Zero Zero Point, Three Pole Points

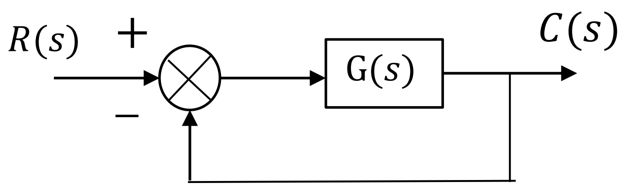
Ex.2 $G(s) = \dfrac{K(1 + b_1 s)}{1 + a_1 s + a_2 s^2}$ One Zero Point, Two Pole Points

Ex.3 $G(s) = \dfrac{K(1 + b_1 s)}{1 + a_1 s + a_2 s^2 + a_3 s^3}$ One Zero Point, Three Pole Points

Ex.4 $G(s) = \dfrac{K(1 + b_1 s + b_2 s^2)}{1 + a_1 s + a_2 s^2 + a_3 s^3}$ Two Zero Points, Three Pole Points

Based on Routh-Hurwitz Criterion

Example 3



Open-loop transfer function:

$$G(s) = \frac{K(1+bs)}{1 + a_1 s + a_2 s^2 + a_3 s^3}$$

Closed-loop transfer function:

$$H(s) = \frac{G(s)}{1+G(s)} = \frac{K + Kbs}{1 + K + (a_1 + Kb)s + a_2 s^2 + a_3 s^3}$$

Routh table

S^3	a_3	$a_1 + Kb$
S^2	a_2	$1 + K$
S^1	$\frac{a_2(a_1 + Kb) - a_3(1+K)}{a_2}$	
S^0	$1 + K$	

Based on Routh-Hurwitz criterion:

$a_3 > 0$ $\quad$ $a_2 > 0$

$1 + K > 0$

$$\frac{a_2(a_1 + Kb) - a_3(1+K)}{a_2} > 0 \Rightarrow \begin{cases} K > \dfrac{a_3 - a_1 a_2}{a_2 b - a_3} \\ K < \dfrac{a_3 - a_1 a_2}{a_2 b - a_3} \end{cases}$$

At condition: $a_2 b - a_3 > 0$

At condition: $a_2 b - a_3 < 0$

Based on Nyquist Criterion

Frequency domain:

$$G(j\omega) = \frac{K(1+bj\omega)}{1-a_2\omega^2+j(a_1\omega-a_3\omega^3)} = \frac{K[(1-a_2\omega^2+a_1b\omega^2-a_3b\omega^4)+j(b\omega-a_2b\omega^3-a_1\omega+a_3\omega^3)]}{(1-a_2\omega^2)^2+(a_1\omega-a_3\omega^3)^2}$$

Special frequency expressions

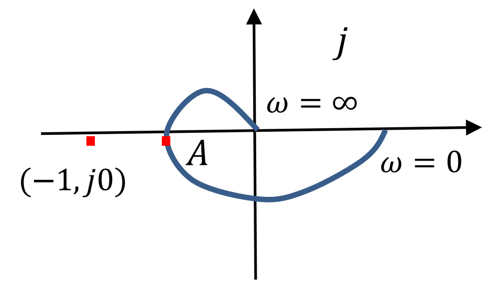
$$\angle G(j\omega) = -\pi$$

$$\Rightarrow \quad b\omega - a_2b\omega^3 - a_1\omega + a_3\omega^3 = 0$$

$$\Rightarrow \quad \omega^2 = \frac{a_1-b}{a_3-a_2b}$$

At point A

$$\Rightarrow \quad |G(j\omega)| = \left|\frac{K(1-a_2\omega^2+a_1b\omega^2-a_3b\omega^4)}{(1-a_2\omega^2)^2+(a_1\omega-a_3\omega^3)^2}\right| = K\left|\frac{a_3-a_2b}{a_3-a_1a_2}\right|$$



sketch chart of Nyquist plot

Stability condition:

$$|G(j\omega)| < 1 \Rightarrow \begin{cases} \dfrac{a_3-a_1a_2}{a_2b-a_3} < K < \dfrac{a_3-a_1a_2}{a_3-a_2b} & \text{At condition: } (a_3-a_1a_2)(a_3-a_2b) > 0 \\ \dfrac{a_3-a_1a_2}{a_3-a_2b} < K < \dfrac{a_3-a_1a_2}{a_2b-a_3} & \text{At condition: } (a_3-a_1a_2)(a_3-a_2b) < 0 \end{cases}$$

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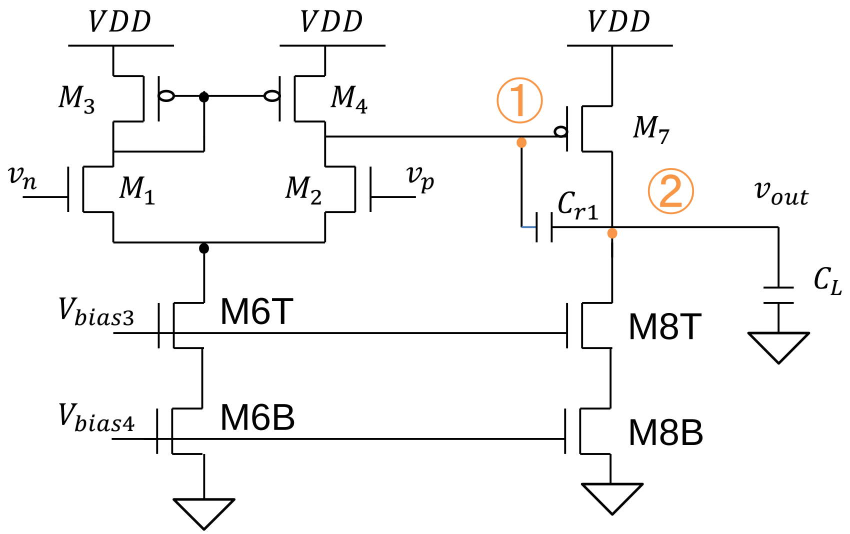
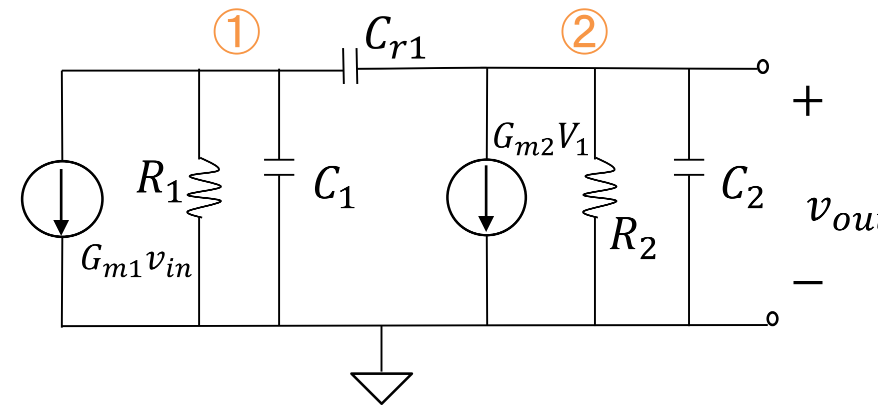
  Ex.1: Two-stage amplifier with C compensation

  Ex.2: Two-stage amplifier with C, R compensation
- Simulation Verification
- Discussion & Conclusion

Amplifier 1



Transistor level circuit



Small-signal model

Fig.1 Two-stage amplifier with inter-stage capacitance

Open-loop transfer function from small signal model

$$A(s) = \frac{v_{out}(s)}{v_{in}(s)} = A_0 \frac{1 + b_1 s}{1 + a_1 s + a_2 s^2}$$

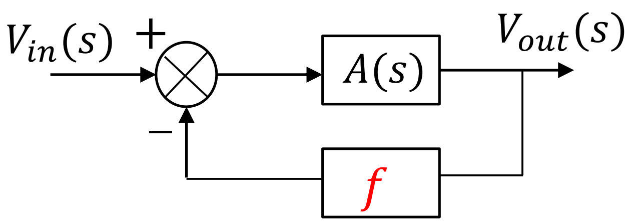
$$b_1 = -\frac{C_r}{G_{m2}} \qquad A_0 = G_{m1} G_{m2} R_1 R_2 \qquad v_{in} = v_p - v_n$$

$$a_1 = R_1 C_1 + R_2 C_2 + (R_1 + R_2 + R_1 G_{m2} R_2) C_r \qquad a_2 = R_1 R_2 (C_1 C_2 + C_1 C_r + C_2 C_r)$$

Routh-Hurwitz method

Closed-loop transfer function:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{A(s)}{1 + fA(s)} = \frac{A_0(1 + b_1 s)}{1 + fA_0 + (a_1 + fA_0 b_1)s + a_2 s^2}$$



Closed-loop configuration

Explicit stability condition of parameters:

$$\theta = a_1 + fA_0 b_1$$
$$= R_1 C_1 + R_2 C_2 + (R_1 + R_2)C_r + (G_{m2} - fG_{m1})R_1 R_2 C_r > 0$$

θ: time dimension parameter

Relationship: θ and phase margin

MATLAB

Data fitting

Short-channel CMOS parameters:

$R_1 = r_{on}||r_{op} = 111k\Omega$

$R_2 = r_{op}||R_{ocasn} \approx r_{op} = 333k\Omega$

$G_{m1} = g_{mn} = 100\,uA/V$

$G_{m2} = g_{mp} = 180\,uA/V$

$C_1 = C_{dg4} + C_{dg2} + C_{gs7} = 13.6fF$

$C_2 = C_L + C_{gd8} \approx C_L + 1.56fF$
$= 101.56fF \quad (C_L = 100fF)$

Data Processing by MATLAB

- Data collection: [GM, PM, F_{gm}, F_{pm}]=margin(G)

f=0.01										
C_{r1} [fF]	10	20	30	40	50	60	70	80	90	…
θ [uS]	0.11	0.18	0.25	0.32	0.39	0.46	0.53	0.60	0.67	…
PM [degree]	16	19	22	24	27	29	31	33	34	…
GM [dB]	9.1	7.6	7.0	6.6	6.4	6.3	6.2	6.0	6.0	…
F_{gm} [GHz]	4.5	3.4	2.9	2.6	2.3	2.1	2.0	1.9	1.8	…
F_{pm} [GHz]	2.6	2.1	1.8	1.5	1.4	1.2	1.1	1.0	9.4	…

- Data fitting： p=polyfit(x,y,n) Curve Fitting Tool

Data Fitting Result

Fitted Curve

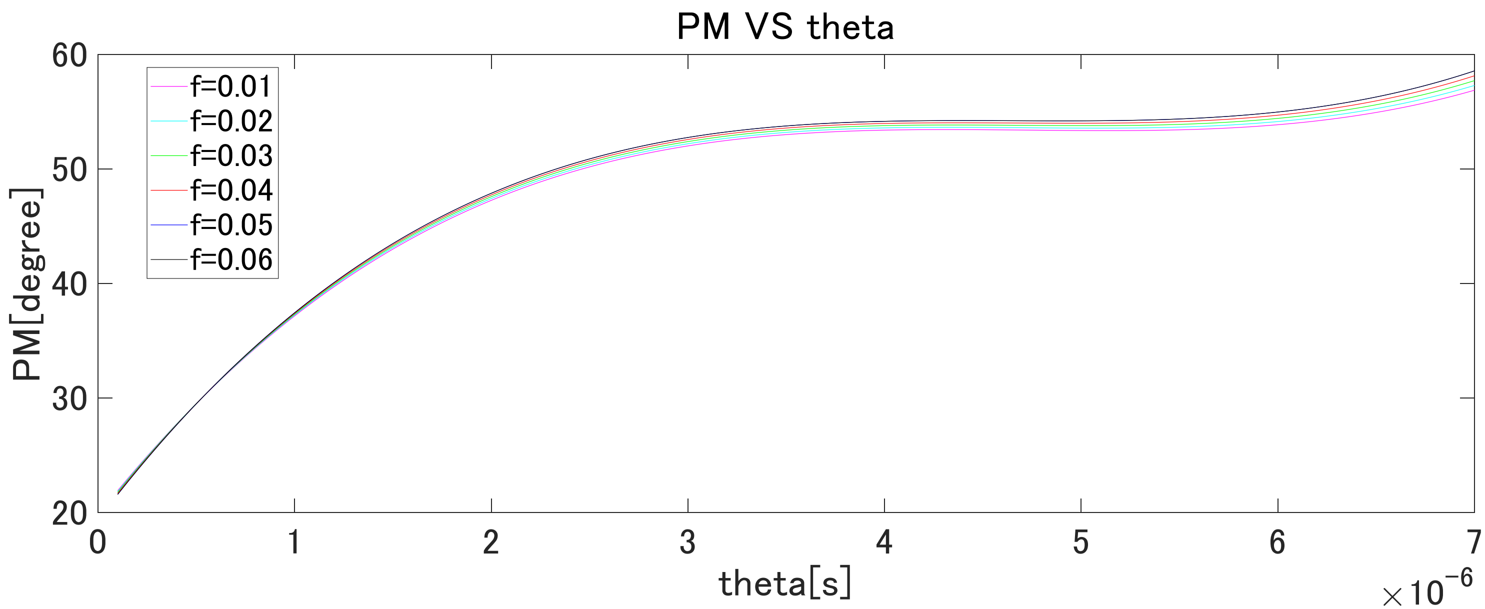


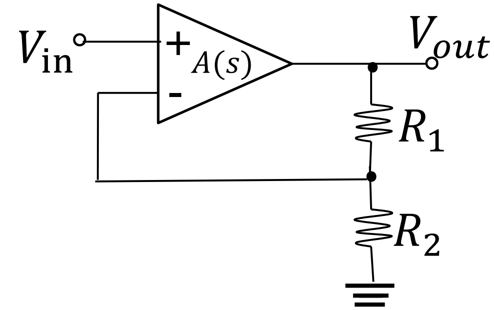
Fig.2 Relationship between PM and parameter θ at various feedback factor conditions.

- One-to-one relationship
- increase of parameter’s value

  ⇒ phase margin will be increased

  feedback system will be more stable

f =0.01 Condition



Relation function:

PM= $f_1(\theta)$

$$= 2.601e^{28}\theta^5 - 5.616e^{23}\theta^4 + 4.683e^{18}\theta^3 - 1.915e^{13}\theta^2 + 4.076e^{28}\theta + 13.38$$

θ: independent variable

PM: dependent variable

$$f = \frac{R_2}{R_1 + R_2}$$

線形モデル Poly5:

f(x) = p1*x^5 + p2*x^4 + p3*x^3 + p4*x^2 + p5*x + p6

係数 (95% の信頼限界):

p1 = 2.601e+28 (2.297e+28, 2.904e+28)
p2 = -5.616e+23 (-6.164e+23, -5.067e+23)
p3 = 4.683e+18 (4.324e+18, 5.043e+18)
p4 = -1.915e+13 (-2.018e+13, -1.811e+13)
p5 = 4.076e+07 (3.953e+07, 4.199e+07)
p6 = 13.38 (12.93, 13.83)

適合度:

SSE: 9.595
決定係数: 0.9987
自由度調整済み決定係数: 0.9986
RMSE: 0.3195

Curve Fitting Tool

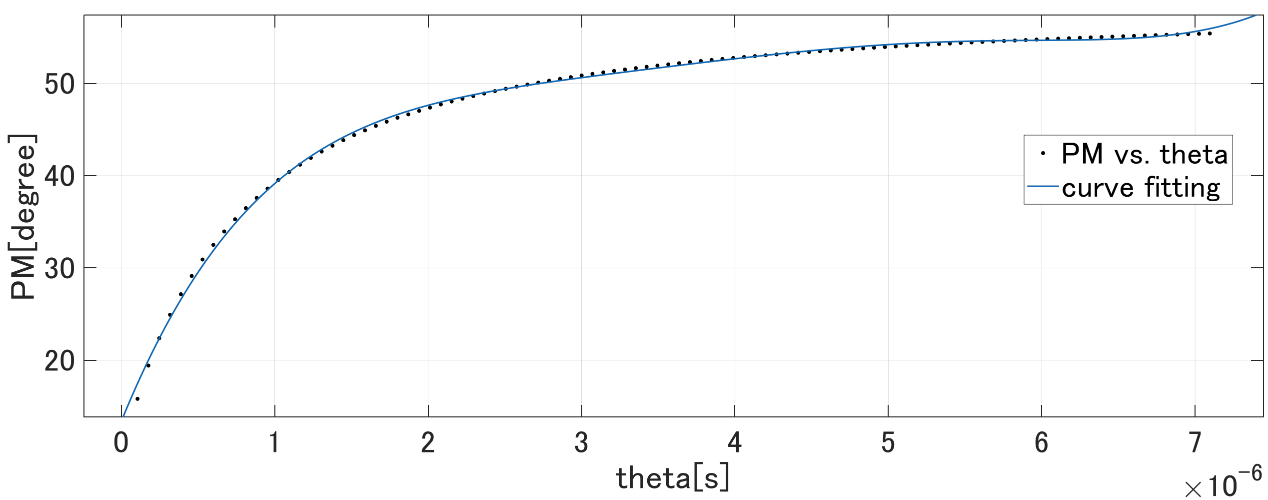
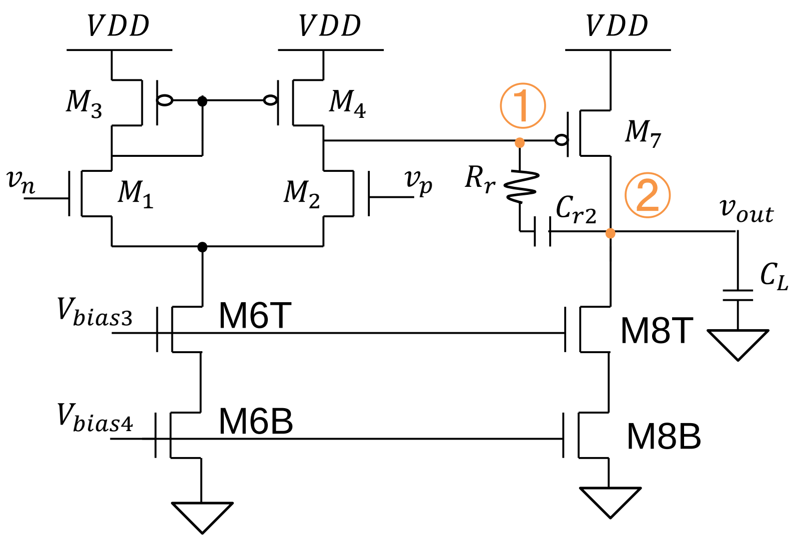
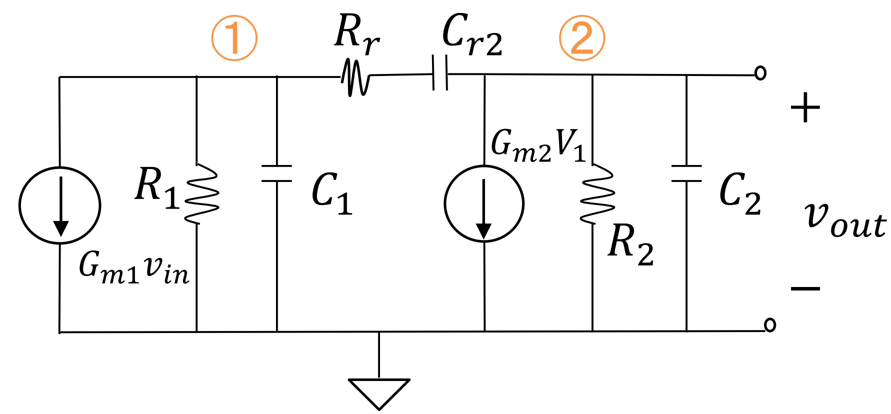


Fig.3 Relationship between PM with parameter θ at feedback factor $f = 0.01$ condition.

Amplifier 2



(a) Transistor level circuit



(b) Small-signal model

Fig.4 Two-pole amplifier with compensation network using a nulling resistor

Open-loop transfer function:

$$A(s) = \frac{v_{out}(s)}{v_{in}(s)} = A_0 \frac{1 + d_1 s}{1 + a_1 s + a_2 s^2 + a_3 s^3}$$

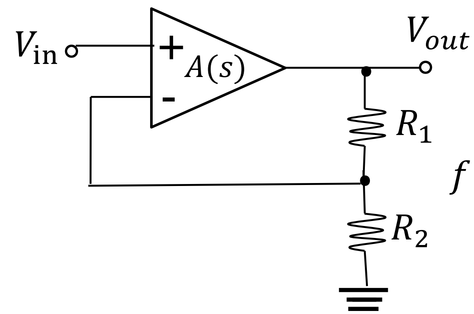
$A_0 = G_{m1} G_{m2} R_1 R_2$ $\quad d_1 = -\left(\frac{C_r}{G_{m2}} - R_r C_r\right)$ $\quad a_1 = R_1 C_1 + R_2 C_2 + (R_1 + R_2 + R_r + R_1 R_2 G_{m2}) C_r$

$a_2 = R_1 R_2 (C_2 C_r + C_1 C_2 + C_1 C_r) + R_r C_r (R_1 C_1 + R_2 C_2)$ $\quad v_{in} = v_p - v_n$

Routh-Hurwitz Method

Closed-loop transfer function:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{A(s)}{1+fA(s)} = \frac{A_0(1+d_1 s)}{1+fA_0+(a_1+fA_0d_1)s+a_2s^2+a_3s^3}$$



$$f = \frac{R_2}{R_1+R_2}$$

Explicit stability condition of parameters:

$$\alpha = a_1 + fA_0d_1$$

$$= R_1C_1 + R_2C_2 + (R_1+R_2+R_r)C_r + (G_{m2} - fG_{m1} + fG_{m1}G_{m2}R_r)R_1R_2C_r > 0$$

$$\beta = \frac{(a_1+fA_0d_1)a_2 - a_3(1+fA_0)}{a_2} > 0$$

$(parameter\ of\ Routh\ stable)$

α, β: time dimension parameters

Routh table

s^n	α_n	α_{n-2}	α_{n-4}	α_{n-6}	$\cdots$
s^{n-1}	α_{n-1}	α_{n-3}	α_{n-5}	α_{n-7}	$\cdots$
s^{n-2}	$\beta_1 = \frac{\alpha_{n-1}\alpha_{n-2} - \alpha_n\alpha_{n-3}}{\alpha_{n-1}}$	$\beta_2 = \frac{\alpha_{n-1}\alpha_{n-4} - \alpha_n\alpha_{n-5}}{\alpha_{n-1}}$	β_3	β_4	$\cdots$
s^{n-3}	$\gamma_1 = \frac{\beta_1\alpha_{n-3} - \alpha_{n-1}\beta_2}{\beta_1}$	$\gamma_2 = \frac{\beta_1\alpha_{n-5} - \alpha_{n-1}\beta_3}{\beta_1}$	γ_3	γ_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	α_0				

Relationship: α, β and phase margin

Interpolation by MATLAB

Data Collection

$$C_{r1}\begin{cases} R_{r11} & (\alpha_{11},\beta_{11}) \\ R_{r12} & (\alpha_{12},\beta_{12}) \\ R_{r13} & (\alpha_{13},\beta_{13}) \\ \cdots & \cdots \\ R_{r19} & (\alpha_{19},\beta_{19}) \end{cases}
\qquad
C_{r2}\begin{cases} R_{r21} & (\alpha_{21},\beta_{21}) \\ R_{r22} & (\alpha_{22},\beta_{22}) \\ R_{r23} & (\alpha_{23},\beta_{23}) \\ \cdots & \cdots \\ R_{r29} & (\alpha_{29},\beta_{29}) \end{cases}$$

$$C_{r3}\begin{cases} R_{r31} & (\alpha_{31},\beta_{31}) \\ R_{r32} & (\alpha_{32},\beta_{32}) \\ R_{r33} & (\alpha_{33},\beta_{33}) \\ \cdots & \cdots \\ R_{r39} & (\alpha_{39},\beta_{39}) \end{cases}
\quad \cdots \quad
C_{r9}\begin{cases} R_{r91} & (\alpha_{91},\beta_{91}) \\ R_{r92} & (\alpha_{92},\beta_{92}) \\ R_{r93} & (\alpha_{93},\beta_{93}) \\ \cdots & \cdots \\ R_{r99} & (\alpha_{99},\beta_{99}) \end{cases}$$

Produce $9 * 9 = 81$ groups data

Interpolation by MATLAB

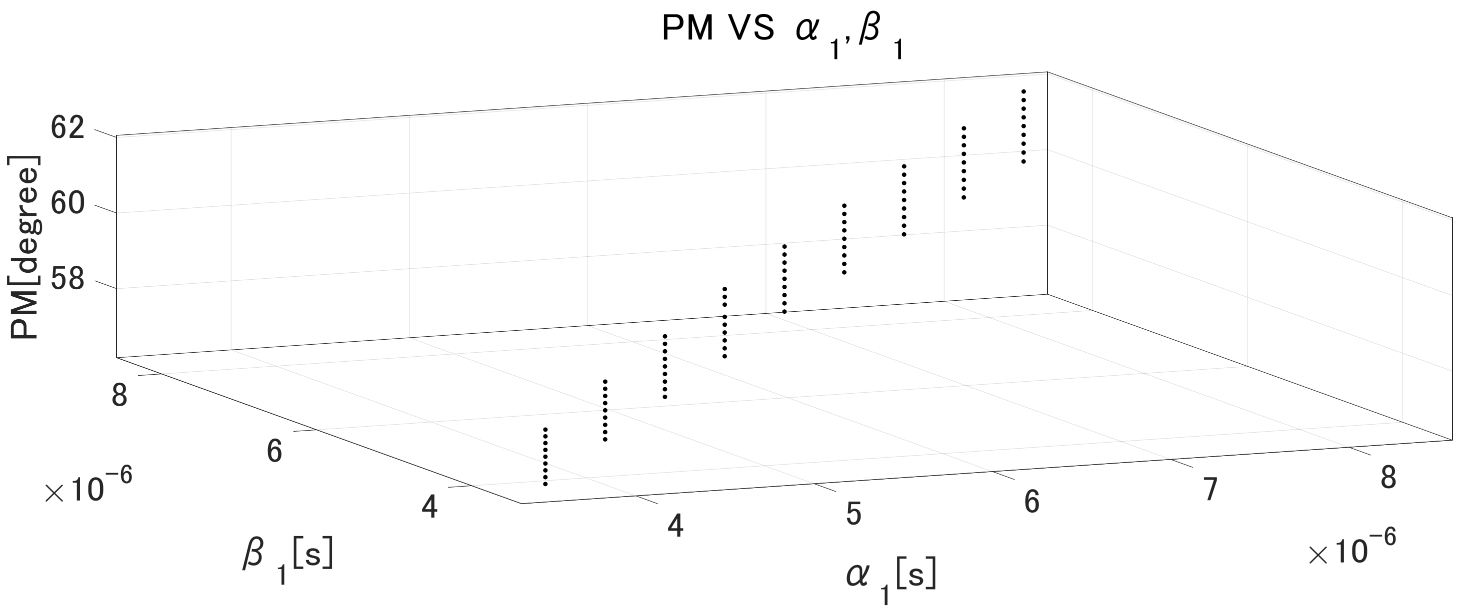


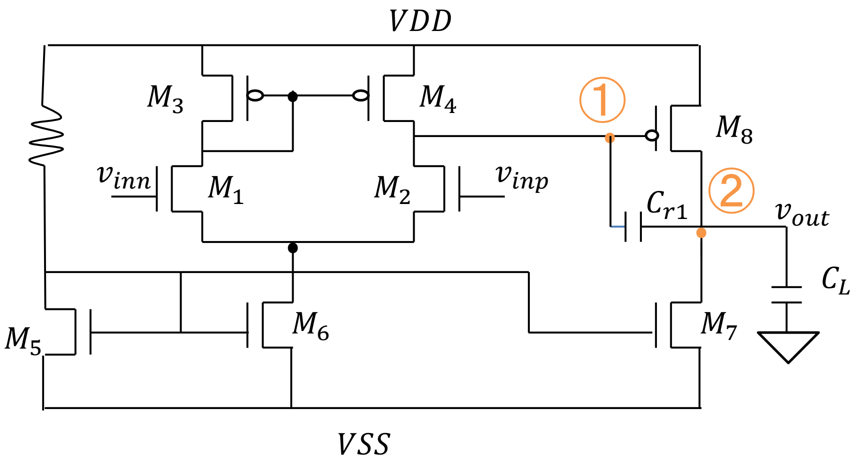
Fig.5 Relationship between PM with parameter α_1, β_1
at feedback factor $f = 0.01$ condition.

- Linear relationship
- increase of parameter's value

  phase margin will be increased
  feedback system will be more stable

Contents

Verification Circuit



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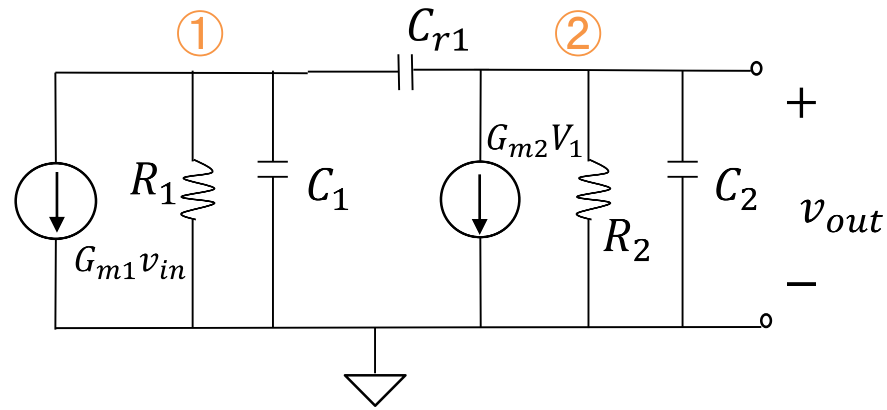
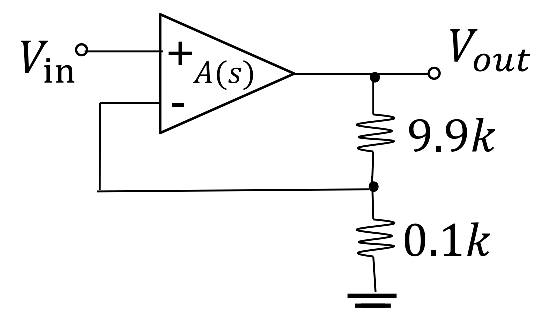


(b) Small-signal model

Fig.6 Two-pole amplifier with inter-stage capacitance



Closed-loop transfer function:

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{A(s)}{1 + fA(s)} = \frac{A_0(1 + b_1 s)}{1 + fA_0 + (a_1 + fA_0 b_1)s + a_2 s^2}$$

$$f = \frac{0.1}{0.1 + 9.9} = 0.01$$

Explicit stability condition of parameters:

$$\theta = a_1 + fA_0 b_1$$

$$= R_1 C_1 + R_2 C_2 + (R_1 + R_2) C_{r1} + (G_{m2} - f G_{m1}) R_1 R_2 C_{r1} > 0$$

Data Fitting by MATLAB

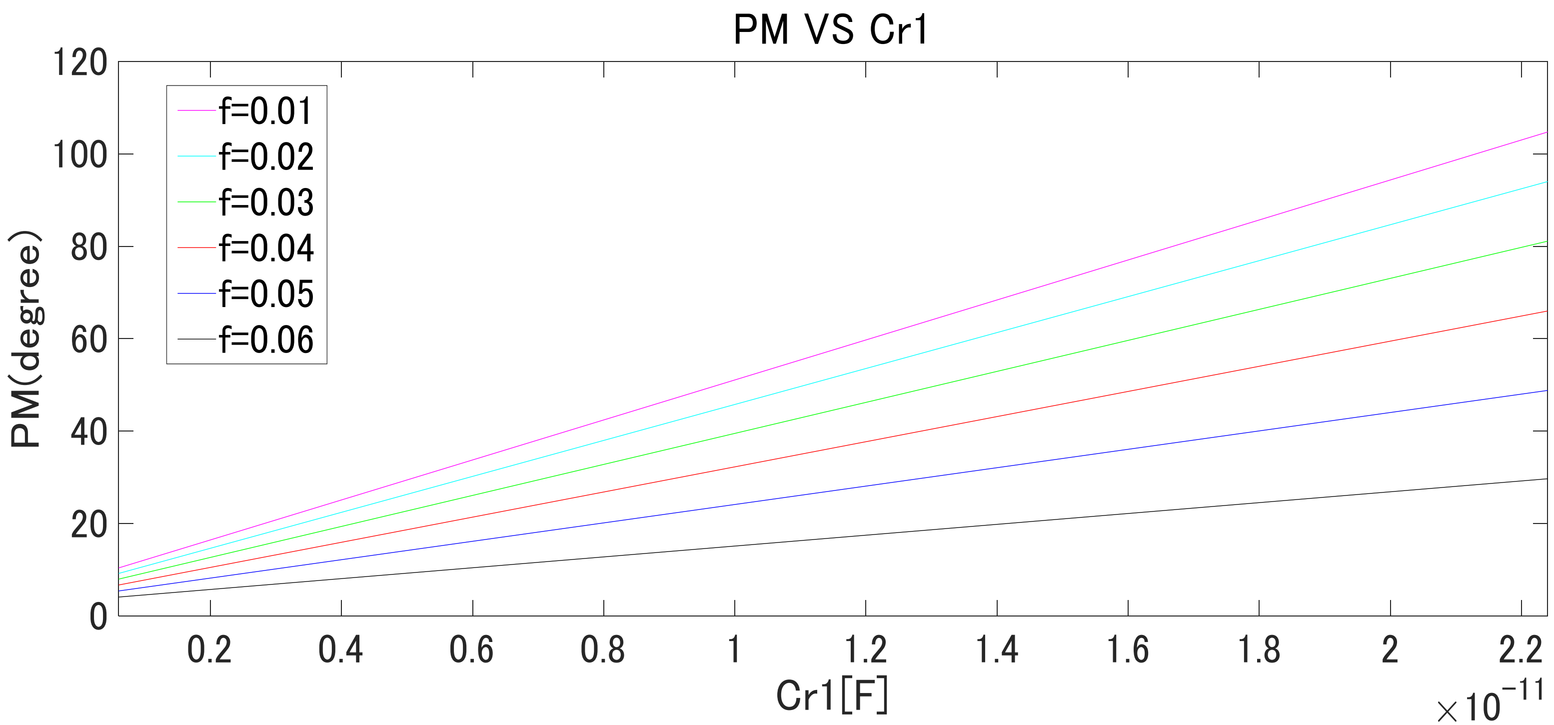
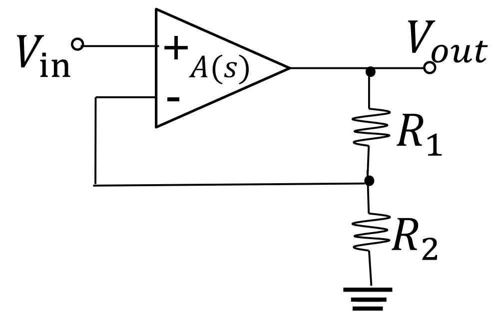


Fig.7 Relationship between PM with compensation capacitor C_{r1} at variation feedback factor f conditions.

PM versus C_{r1}

f = 0.01 condition



Relation function:

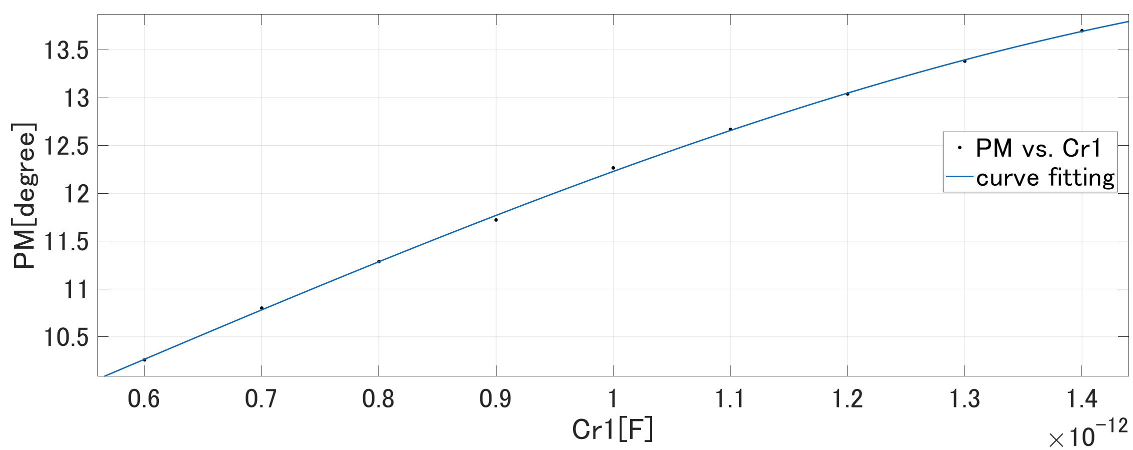
$$PM = f_1(C_{r1})$$
$$= -1.026e^{36}{C_{r1}}^{3} + 1.52e^{24}{C_{r1}}^{2} + 4.488e^{12}C_{r1} + 7.247$$

$$f = \frac{R_2}{R_1 + R_2} = 0.01$$

C_{r1}: independent variable
PM: dependent variable

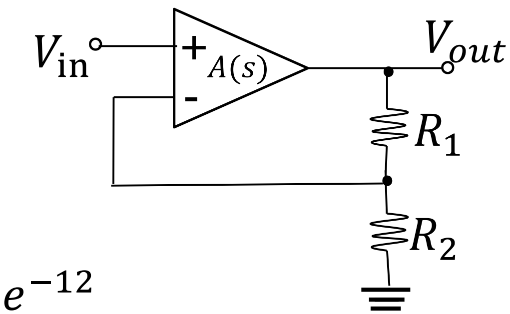
線形モデル Poly3:
f(x) = p1*x^3 + p2*x^2 + p3*x + p4
係数 (95% の信頼限界):
p1 = -1.026e+36 (-3.052e+36, 9.994e+35)
p2 = 1.52e+24 (-4.573e+24, 7.612e+24)
p3 = 4.488e+12 (-1.415e+12, 1.039e+13)
p4 = 7.247 (5.412, 9.083)

適合度:
SSE: 0.004426
決定係数: 0.9996
自由度調整済み決定係数: 0.9994
RMSE: 0.02975



C_{r1} versus PM

f = 0.01 condition



Relation function:

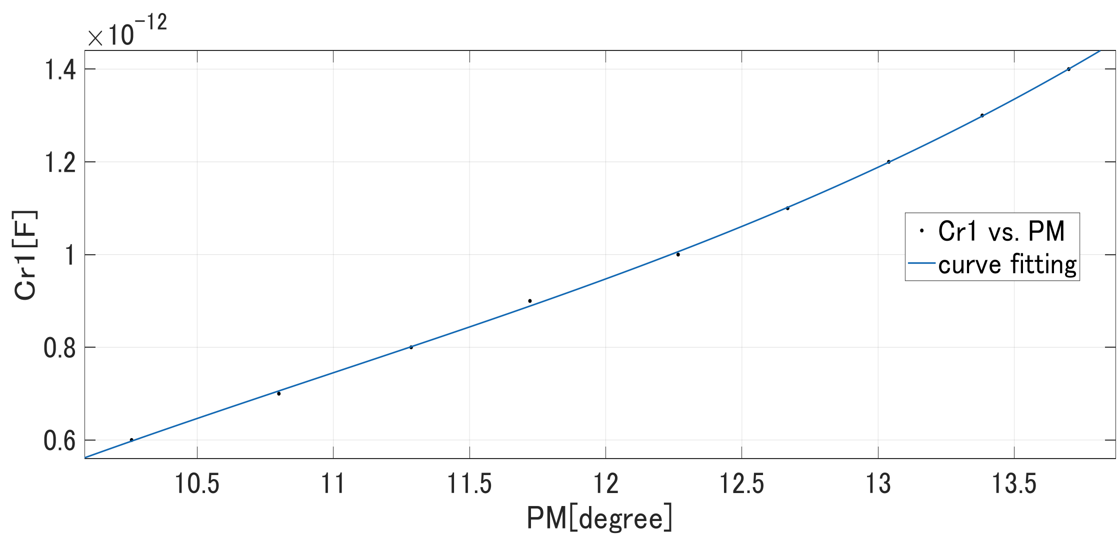
$$C_{r1} = f_1(PM)$$
$$= 6.343e^{-15}PM^3 - 2.091e^{-13}PM^2 + 2.493e^{-12}PM - 9.822e^{-12}$$

C_{r1}: dependent variable
PM: independent variable

$$f = \frac{R_2}{R_1 + R_2} = 0.01$$

線形モデル Poly3:
　f(x) = p1*x^3 + p2*x^2 + p3*x + p4
係数 (95% の信頼限界):
　p1 = 6.343e-15 (8.692e-16, 1.182e-14)
　p2 = -2.091e-13 (-4.059e-13, -1.223e-14)
　p3 = 2.493e-12 (1.435e-13, 4.843e-12)
　p4 = -9.822e-12 (-1.913e-11, -5.162e-13)

適合度:
　SSE: 2.085e-28
　決定係数: 0.9997
　自由度調整済み決定係数: 0.9994
　RMSE: 6.457e-15



Practicability

For stable feedback system,
necessary PM value: 45 degree or 60 degree

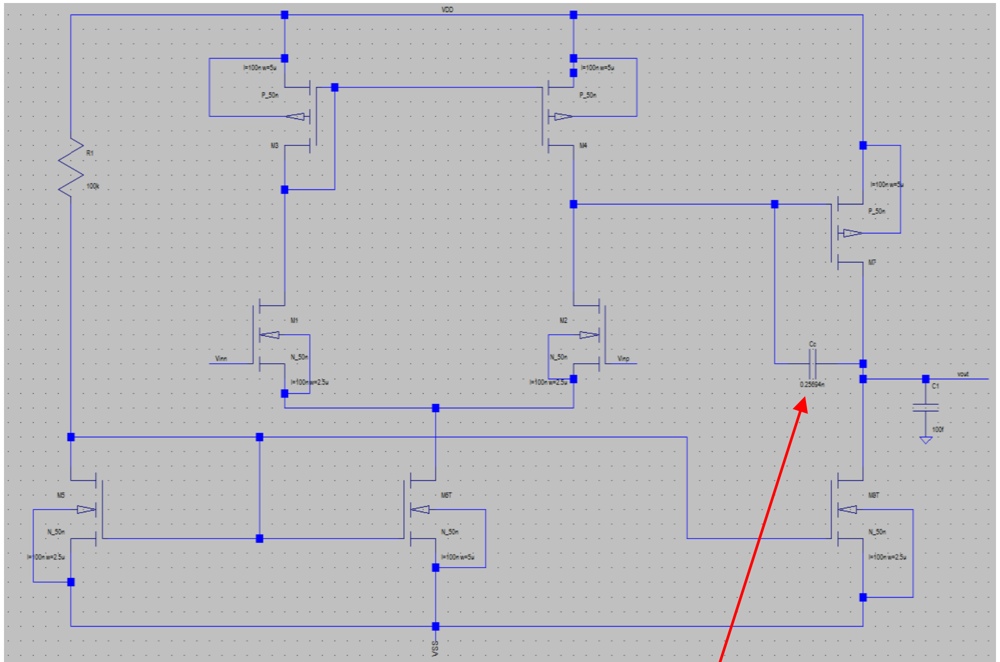
$$C_{r1} = f_1(PM)$$
$$= 6.343e^{-15}PM^3 - 2.091e^{-13}PM^2 + 2.493e^{-12}PM - 9.822e^{-12}$$

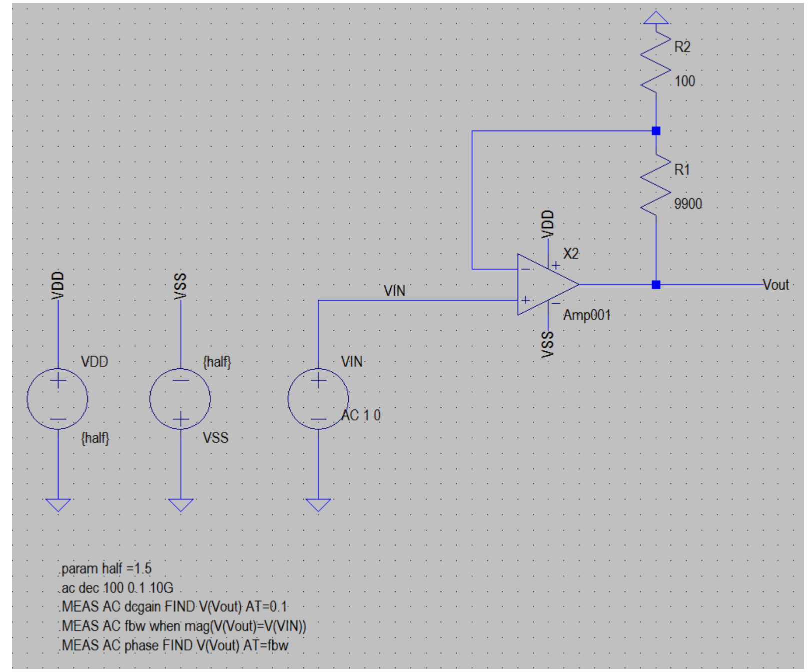
PM=45degree, $C_{r1} = 2.5694e^{-10}F = 0.25694\mathrm{nF}$

PM=60degree, $C_{r1} = 7.5709e^{-10}F = 0.75709\mathrm{nF}$

For stability and needed PM value,
compensation capacitance can be calculated.

Simulation by LTspice





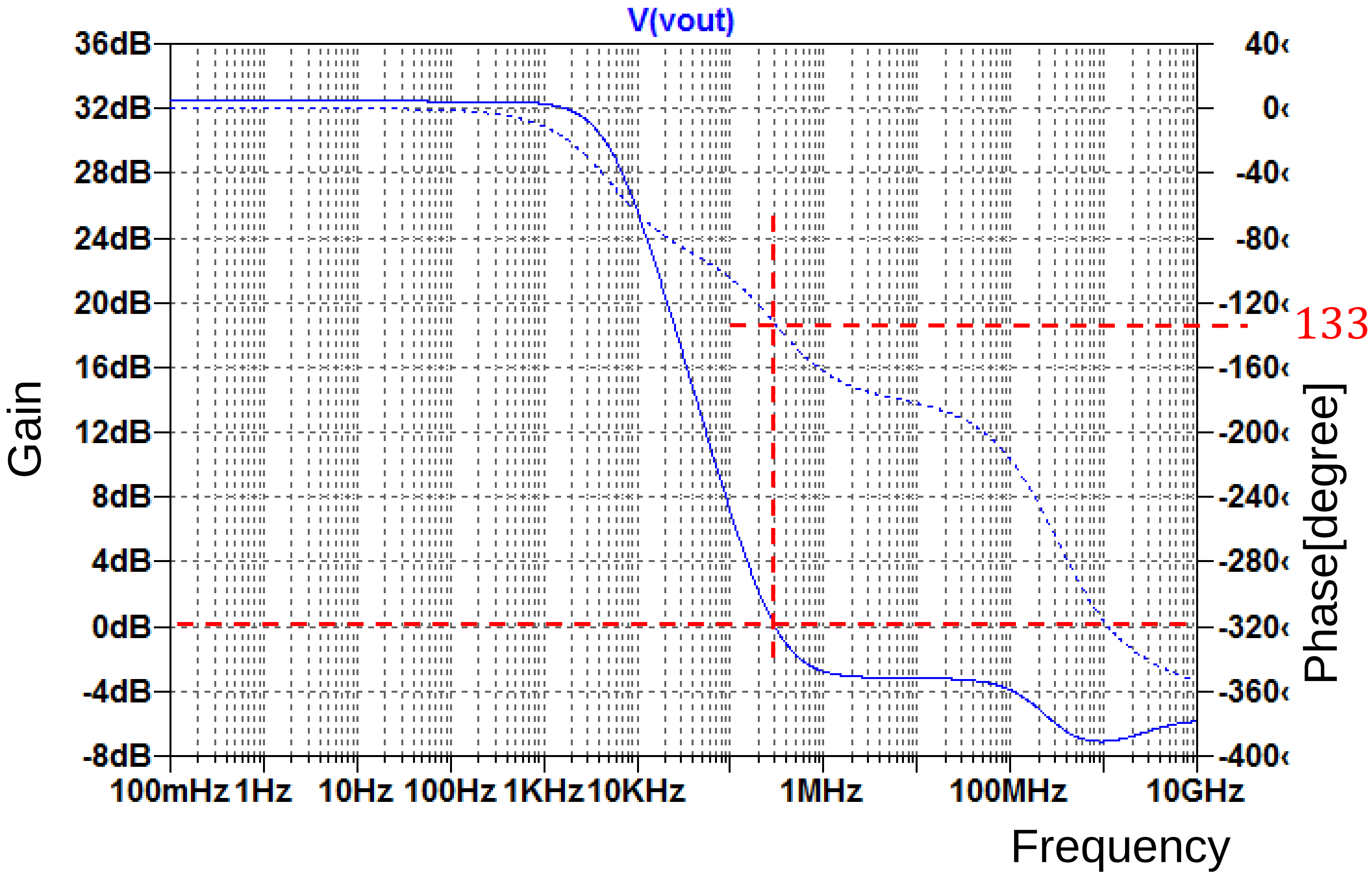
compensation capacitor:

$C_{r1} = 0.25694nF$

feedback factor:

$$f = \frac{0.1k}{9.9k} = 0.01$$

Simulation Result




```
phase: v(vout)=(-4.66844e-005dB,-133.013°) at 301437
```

$$Phase\ Margin = 180^{\circ} - 133^{\circ} = 47^{\circ}$$

Contents

Discussion

Depict small signal equivalent circuit of amplifier

↓

Derive open-loop transfer function

↓

Derive closed-loop transfer function
& obtain characteristics equation

↓

Apply R-H stability criterion
& obtain explicit stability condition

Especially effective for

Multi-stage opamp (high-order system)

Limitation

Explicit transfer function with polynomials of s has to be derived.

Conclusion

- R-H method, explicit circuit parameter conditions can be obtained for feedback stability.
- Equivalency of their mathematical foundations was shown
- Relationship between R-H criterion parameter with PM:
  - linear relationship
  - the system will be more stable, following with the increase of parameter's value.
- The proposed method has been confirmed with LTspice simulation

R-H method can be used
with conventional Bode plot method.

Dr. Yuji Gendai is acknowledged
for his suggestions and helpful comments

Thank you
for your kind attention.