Analog Signal Generator for Irrational Number Approximation Based on Number Theory

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Outline

• Research objective
• R-r resistor ladder
  – Convergence resistance value
  – Metallic mean and $\sqrt{2}$ approximation ladder
• Resistor ladder with different resistance values
  – Correspondence
    • combined resistance and continued fraction
• Resistor network digital-to-analog converters
• Conclusion
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Research Objective

- On integrated circuit, resistance absolute value $\rightarrow$ vary resistance ratio $\rightarrow$ accurate

- Irrational number $\Leftrightarrow$ continued fraction configured by integers

- By connecting resistors with integer ratio $\rightarrow$ irrational number approximation ratio

- Generate irrational number approximation analog signal
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R-r Resistor Ladder

R-r resistor ladder network

Unit resistive ladder
Combined Resistance Value

- Increase the number of stages

\[ Z_2 = R + \frac{r(R + r)}{r + (R + r)} \]

\[ Z_{k+1} = R + \frac{rZ_k}{r + Z_k} \]

\[ = \frac{(r + R)Z_k + rR}{Z_k + r} \]

Recurrence relation of \( Z_k \)
Combined Resistance Value

\[ Z_k = \frac{\alpha \gamma^k - \beta}{\gamma^k - 1} \]

Here, \(\alpha = \frac{1}{2} \left( R + \sqrt{R^2 + 4rR} \right)\), \(\beta = \frac{1}{2} \left( R - \sqrt{R^2 + 4rR} \right)\), \(\gamma = \frac{R + r - \beta}{R + r - \alpha}\), \(1 < \gamma\)

Convergence value:

\[ Z_\infty = \frac{R}{2} + \frac{\sqrt{R(R + 4r)}}{2} \]
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Metallic Mean $\lambda$

- Positive root of
  \[ x^2 - nx - 1 = 0 \]
  \[ \downarrow \]
  \[ \lambda_n = \frac{n}{2} + \frac{\sqrt{n^2 + 4}}{2} \]

- Continued fraction expansion
  \[ \lambda_n = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \cdots}}}. \]

- $n = 1$: golden ratio $\phi$
  \[ \phi = \frac{1 + \sqrt{5}}{2} \]

- $n = 2$: silver mean $\tau$
  \[ \tau = 1 + \sqrt{2} \]

- $n = 3$: bronze mean $\xi$
  \[ \xi = \frac{3 + \sqrt{13}}{2} \]
R-r Ladder and Metallic Means

Resistance value of R-r ladder

\[ Z_\infty = \frac{R}{2} + \frac{\sqrt{R(R + 4r)}}{2} \]

\[ Z_{k+1} = R + \frac{rZ_k}{r + Z_k} \]

\[ = \frac{R}{m} \left( m + \frac{1}{\frac{R}{mr} + \frac{R}{mZ_k}} \right) \]

\[ = \frac{R}{m} \left( m + \frac{1}{\frac{R}{mr} + \frac{1}{m + \frac{1}{\frac{R}{mr} + \frac{1}{\ddots}}} \right) \]

Metallic mean

\[ \lambda_n = \frac{n}{2} + \frac{\sqrt{n^2 + 4}}{2} \]

\[ \lambda_n = n + \frac{1}{n + \frac{1}{\ddots}} \]

Combined resistance of R-r ladder

Metabolic mean ratio (irrational number)
\[ Z_{R,R} = \frac{R}{2} + \frac{\sqrt{R(R + 4r)}}{2} \]
\[ = \frac{1 + \sqrt{5}}{2} R \]

Golden ratio \( \phi \) ladder
2R-0.5R Resistor ladder

\[ Z_{2R,0.5R} = \frac{2R}{2} + \frac{\sqrt{2R(2R + 4 \cdot 0.5R)}}{2} \]

\[ = R + \frac{2\sqrt{2R^2}}{2} \]

\[ = (1 + \sqrt{2})R \]

\[ \approx 2.414R \]
2R-0.5R Resistor Ladder (Simulation)

• Simulation conditions
  – $R = 1$ kΩ
  – Supply 1 mA to ladder, calculate $Z_3$ from $V(\text{out})$
• Result
  $$Z_3 = 2.4142855 \text{ kΩ}$$
  $(1 + \sqrt{2} = 2.4142135623\ldots)$
2R-0.5R resistor ladder

- Simulation conditions
  - \( R = 1 \) kΩ
  - Supply 1 mA to ladder, calculate \( Z_5 \) from \( V(out) \)

- Result
  \[
  Z_5 = 2.4142135 \text{ kΩ}
  \]
  \[
  (1 + \sqrt{2} = 2.4142135623\ldots)
  \]
\[ \sqrt{2} \text{ Approximation Ladder} \]

- \[ \sqrt{2} = (1 + \sqrt{2}) - 1 \]
  \[= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}} \]

- \[ Z_{2R,0.5R} = (1 + \sqrt{2})R \]
  \[ Z_{2R,0.5R} - R = \sqrt{2}R \]

Replace the first 2R resistor of 2R-0.5R ladder with R.
Verification of $\sqrt{2}$ Ladder, 2-Stage

- Simulation conditions
  - $R = 1$ kΩ
  - Supply 1 mA to ladder, calculate $Z_2$ from $V(\text{out})$
- Result
  $Z_2 = 1.41666666$ kΩ
  ($\sqrt{2} = 1.41421356237309\ldots$)
Verification of $\sqrt{2}$ Ladder, 5-Stage

- **Simulation conditions**
  - $R = 1$ kΩ
  - supply 1 mA to ladder, calculate $Z_5$ from $V(\text{out})$

- **Result**
  
  $Z_5 = 1.4142136$ kΩ

($\sqrt{2} = 1.41421356237309\ldots$)
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R-ladder with Different Resistance Values

Registence value of k-th stage → weighting by $p_k$ and $q_k$

$$Z_k = p_k R + \frac{q_k R \cdot Z_{k-1}}{q_k R + Z_{k-1}}$$
R-ladder with Different Resistance Value

\[ Z_k = p_k R + \frac{q_k R \cdot Z_{k-1}}{q_k R + Z_{k-1}} \]

\[ = R \left( p_k + \frac{1}{\frac{1}{q_k} + \frac{R}{Z_{k-1}}} \right) \]

\[ = R \left( p_k + \frac{1}{\frac{1}{q_k} + \frac{1}{p_{k-1} + \frac{1}{\frac{1}{q_{k-1}} + \ddots}}} \right) \]

Adjust \( p_k \) and \( q_k \) according to continued fraction of specified number

\[ \text{Resistance ratio to } R \text{ is specified number} \]
Napier's Constant

- Irrational number
- Denoted by $e$
- Natural logarithm
- Continued fraction
  → regularity

\[
e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \ddots}}}} = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]
\]

\[
p_k \rightarrow \text{odd-numbered terms of integer part} \\
2, 2, 1, 1, 6, \ldots
\]

\[
q_k \rightarrow \text{reciprocals of even-numbered terms} \\
1, 1, 1/4, 1, 1, \ldots
\]
Approximation Ladder, 4-Stage

- Simulation condition
  - $R = 1$ kΩ
  - supply 1 mA to ladder, calculate $Z_4$ from $V(\text{out})$
- Result
  
  $2.7183098$ kΩ

\(e \approx [2; 1, 2, 1, 1, 4, 1, 1]\)
Approximation Ladder, 8-Stage

$e \approx [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1]$

- Simulation condition
  - $R = 1$ kΩ
  - supply 1 mA to ladder, calculate $Z_8$ from $V(\text{out})$

- Result
  - $2.7182816$ kΩ
  - ($e = 2.718281828459536 \ldots$)
\[ \pi \approx 3.14159 \]

\[ = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{7}}}} \]

\[ = [3; 7, 15, 1, 25, 1, 7, 4] \]

- Irrational number
- Ratio of a circle's circumference to diameter
- Continued fraction → no regularity

\( p_k \rightarrow \) odd-numbered terms of integer part
\[ 3, 15, 25, 7 \]

\( q_k \rightarrow \) reciprocals of even-numbered terms
\[ 1/7, 1, 1, 1/4 \]
Simulation conditions:
- \( R = 700 \, \Omega \)
- Supply 1 mA to ladder and \( R \), calculate resistance ratio to \( R \) from the ratio of voltages

Result:
Ratio to \( R \): 3.1415896
(Design value: 3.14159)
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Resistor Network Digital-to-Analog Converters

\[ A_{N-1} \]

\[ A_{N-2} \]

\[ A_2 = R + \frac{1}{\frac{1}{r} + \frac{1}{A_1}} \]

\[ A_1 = R + R_T \]

\[ A_n = R + \frac{1}{\frac{1}{r} + \frac{1}{A_{n-1}}} \]
Resistor Network Digital-to-Analog Converters

\[ V_{OUT}(I_1, I_2, \ldots, I_N, R, r, R_T) = R_T \]

\[
\begin{aligned}
&\cdot I_N \cdot \frac{A_{N-1}}{A_{N-1} + R_T} + I_{N-1} \cdot \frac{r||A_{N-2}}{r||A_{N-2} + A_1} + \frac{r}{A_1 + r} \\
&\cdot I_{N-2} \cdot \frac{r||A_{N-3}}{r||A_{N-3} + A_2} + \frac{r}{A_2 + r}
\end{aligned}
\]
5-bit R-2R DAC

- R-2R DAC
  \[ R \rightarrow R \]
  \[ r \rightarrow 2R \]
  \[ R_T \rightarrow R \]

For all \( n \),
\[ A_n = 2R \]

\[
V_{OUT}(I_1, I_2, I_3, I_4, I_5, R) = R \left( I_5 \cdot \frac{2}{3} + I_4 \cdot \frac{1}{3} + \frac{1}{2} \cdot \left( I_3 \cdot \frac{1}{3} + \frac{1}{2} \cdot \left( I_2 \cdot \frac{1}{3} + \frac{1}{2} \cdot \left( I_1 \cdot \frac{1}{3} \right) \right) \right) \right)
\]

\[
= \frac{1}{3} R \left( 2I_5 + I_4 + \frac{1}{2} I_3 + \frac{1}{4} I_2 + \frac{1}{8} I_1 \right)
\]

→ Currents \( I \) weighted in binary
Examples of Resistor network DAC

- **R-R network DAC**

\[
R \rightarrow R \\
r \rightarrow R \\
R_T \rightarrow R
\]

\[
V_{OUT}(I_1, I_2, I_3, I_4, I_5, R) = R \left( I_5 \cdot \frac{34}{55} + I_4 \cdot \frac{13}{55} + \frac{1}{3} \cdot \left( I_3 \cdot \frac{3}{11} + \frac{3}{8} \cdot \left( I_2 \cdot \frac{16}{55} + \frac{8}{21} \cdot \left( I_1 \cdot \frac{21}{55} \right) \right) \right) \right)
\]

\[
= \frac{1}{55} R \left( 34I_5 + 13I_4 + 5I_3 + 2I_2 + I_1 \right)
\]

When \( I_1 \sim I_5 = (I, -I, 0) \)

→ **Operate as a DAC**
Examples of Resistor network DAC

- Weighted in “ternary”
  \[ R \rightarrow 4R \]
  \[ r \rightarrow 3R \]
  \[ R_T \rightarrow 2R \]

For all \( n \),
\[ A_n = 6R \]

\[
V_{OUT}(I_1, I_2, I_3, I_4, I_5, R) = 2R \left( I_5 \cdot \frac{6}{8} + I_4 \cdot \frac{2}{8} + \frac{1}{3} \cdot \left( I_3 \cdot \frac{2}{8} + \frac{1}{3} \cdot \left( I_2 \cdot \frac{2}{8} + \frac{1}{3} \cdot \left( I_1 \cdot \frac{2}{8} \right) \right) \right) \]
\[ = \frac{1}{2} R \left( 3I_5 + I_4 + \frac{1}{3} I_3 + \frac{1}{9} I_2 + \frac{1}{27} I_1 \right). \]

When \( I_1 \sim I_5 = (I, -I, 0) \),

⇒ Operate as a DAC
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Conclusion

• Clarified
  – R-r ladder network ⇔ some irrational numbers
  – By using continued fraction of specified number, equivalent resistor
    (resistance ratio to $R$ is specified number)
  – Approximation accuracy
    → better, as the number of resistors larger

• Resistor network DAC
  – Generalized DAC using resistor ladder
  – New idea of Non-binary DACs
Q&A

• Q. You showed a ternary DAC on slide 32. Can you design a “Quaternary” DAC?
  A. Probably, we can. But I suppose a “Quaternary” DAC needs more current sources than ternary DAC.

• Q. Can you design the $\sqrt{2}$ weighted DAC?
  A. I’m not sure. Probably we can by using a $1 + \sqrt{2}$ ladder or other configuration of ladder.