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IPS4: Analog/Power Supply Circuits and Their Related Technology

Analog Signal Generator for Irrational Number Approximation Based on Number Theory

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Outline

- Research objective
- R-r resistor ladder
  - – Convergence resistance value
  - – Metallic mean and $\sqrt{2}$ approximation ladder
- Resistor ladder with different resistance values
  - – Correspondence
    - combined resistance and continued fraction
- Resistor network digital-to-analog converters
- Conclusion

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Research Objective

- On integrated circuit,
  resistance absolute value → vary
  resistance ratio→ accurate
- Irrational number
  ⇔continued fraction configured by integers

- By connecting resistors with integer ratio
  → irrational number approximation ratio
- Generate irrational number approximation analog signal

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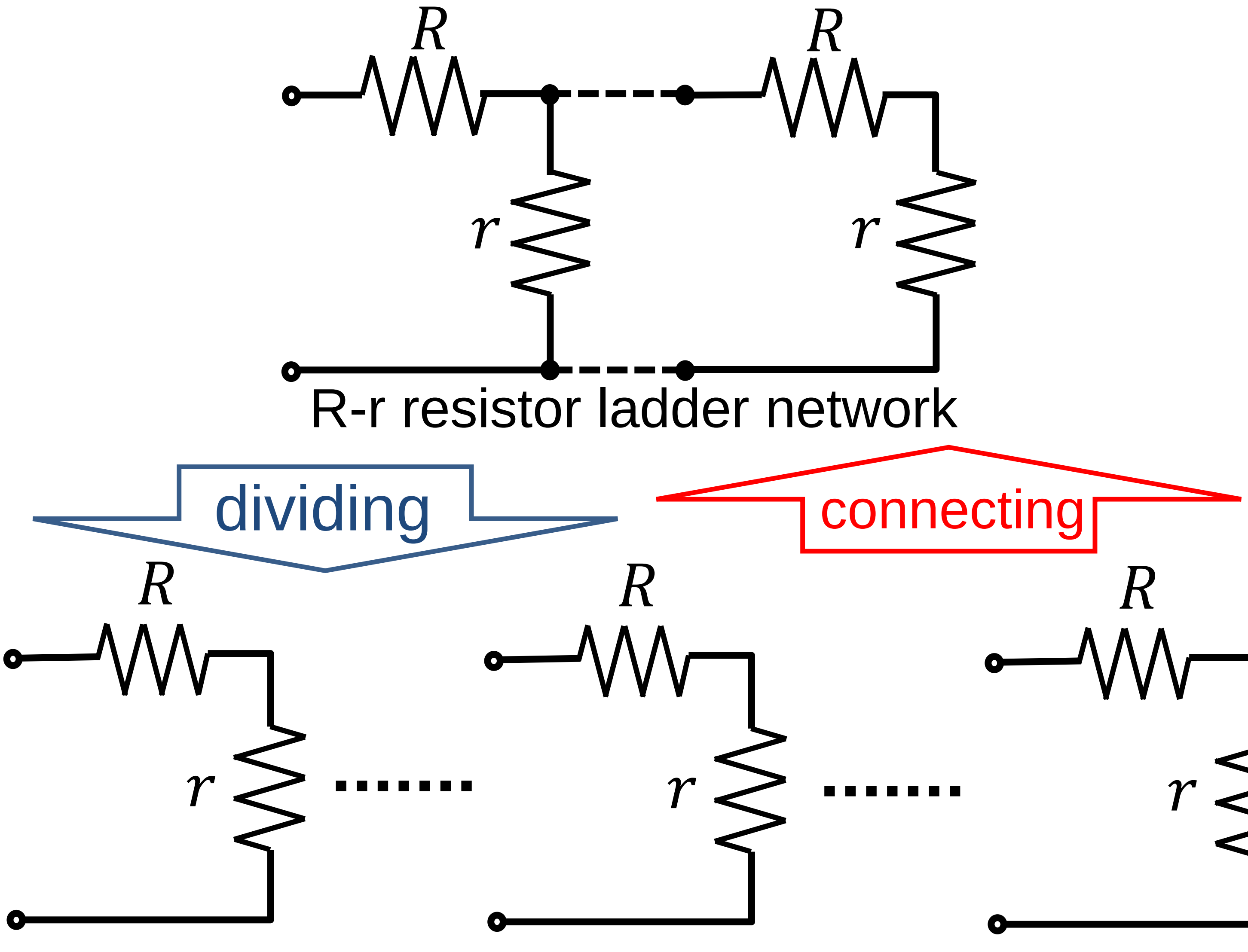
R-r Resistor Ladder

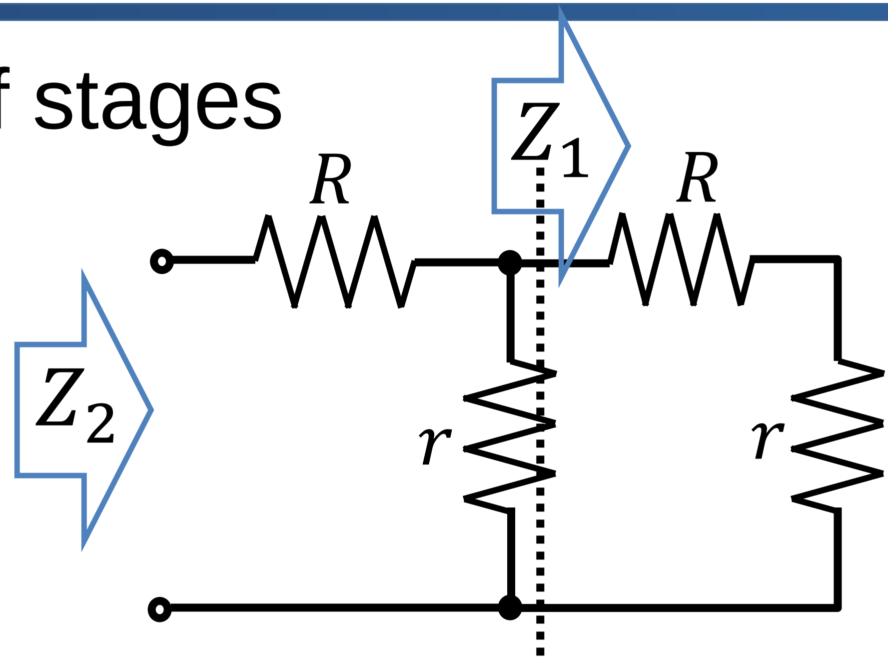


Unit resistive ladder

Combined Resistance Value

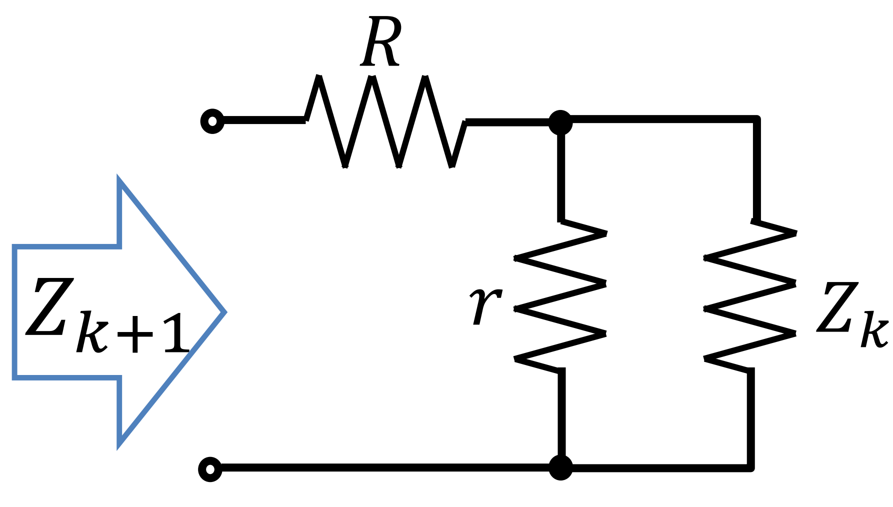
- Increase the number of stages

$$Z_2 = R + \frac{r(R+r)}{r+(R+r)}$$



$$\vdots$$

$$Z_{k+1} = R + \frac{rZ_k}{r+Z_k}$$
$$= \frac{(r+R)Z_k + rR}{Z_k + r}$$



Recurrence relation of Z_k

Combined Resistance Value

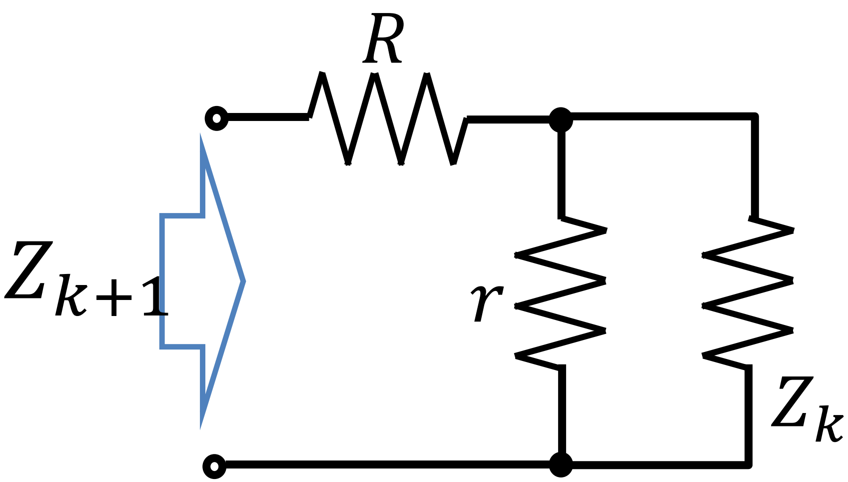
$$Z_k = \frac{\alpha\gamma^k - \beta}{\gamma^k - 1}$$

Here,

$$\alpha = \frac{1}{2}\left(R + \sqrt{R^2 + 4rR}\right),$$

$$\beta = \frac{1}{2}\left(R - \sqrt{R^2 + 4rR}\right),$$

$$\gamma = \frac{R + r - \beta}{R + r - \alpha}, \quad 1 < \gamma$$



Convergence value:

$$Z_\infty = \frac{R}{2} + \frac{\sqrt{R(R + 4r)}}{2}$$

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Metallic Mean λ

- Positive root of

$$x^2 - nx - 1 = 0$$

$$\Downarrow$$

$$\lambda_n = \frac{n}{2} + \frac{\sqrt{n^2+4}}{2}$$

- Continued fraction expansion

$$\lambda_n = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\ddots}}}}$$

- $n = 1$: golden ratio ϕ

$$\phi = \frac{1+\sqrt{5}}{2}$$

- $n = 2$: silver mean τ

$$\tau = 1 + \sqrt{2}$$

- $n = 3$: bronze mean ξ

$$\xi = \frac{3+\sqrt{13}}{2}$$

R-r Ladder and Metallic Means

Resistance value of R-r ladder

$$Z_\infty = \frac{R}{2} + \frac{\sqrt{R(R+4r)}}{2}$$

$$Z_{k+1} = R + \frac{rZ_k}{r+Z_k}$$

$$= \frac{R}{m}\left(m + \frac{1}{\frac{R}{mr} + \frac{R}{mZ_k}}\right)$$

$$= \frac{R}{m}\left(m + \cfrac{1}{\frac{R}{mr} + \cfrac{1}{m + \cfrac{1}{\frac{R}{mr} + \cfrac{1}{\ddots}}}}\right)$$

Metallic mean

$$\lambda_n = \frac{n}{2} + \frac{\sqrt{n^2+4}}{2}$$

$$\lambda_n = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\ddots}}}}$$

Combined resistance of R-r ladder

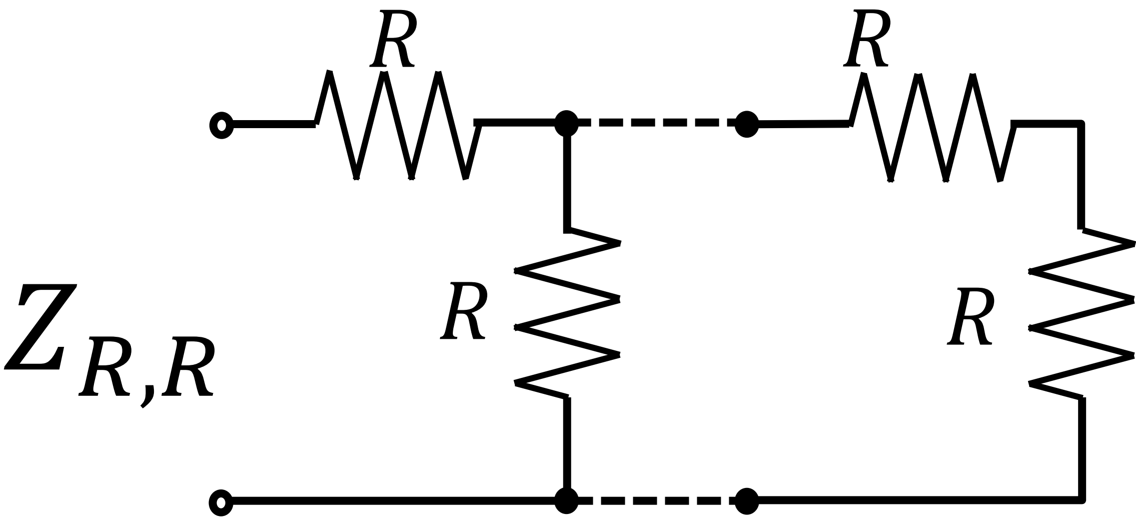
⇨ Metallic mean ratio (irrational number)

R-R Resistor Ladder

$$Z_{R,R} = \frac{R}{2} + \frac{\sqrt{R(R+4r)}}{2}$$

$$= \frac{R}{2} + \frac{\sqrt{R(R+4R)}}{2}$$

$$= \frac{1+\sqrt{5}}{2} R$$



Golden ratio ϕ ladder

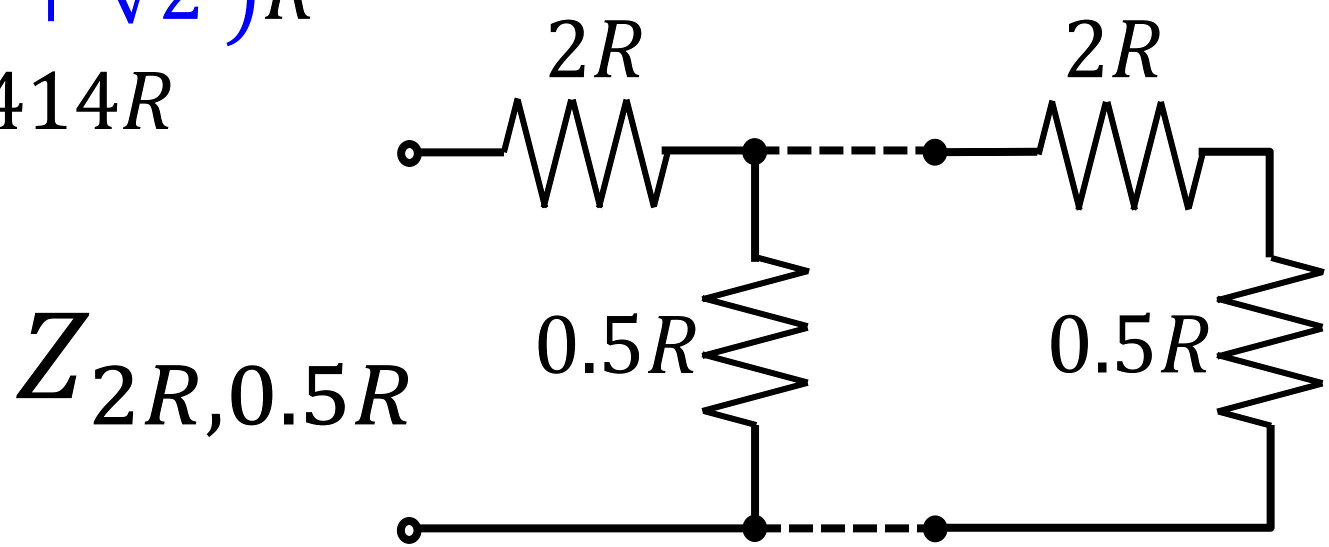
2R-0.5R Resistor ladder

$$Z_{2R,0.5R} = \frac{2R}{2} + \frac{\sqrt{2R(2R + 4 \cdot 0.5R)}}{2}$$

$$= R + \frac{2\sqrt{2R^2}}{2}$$

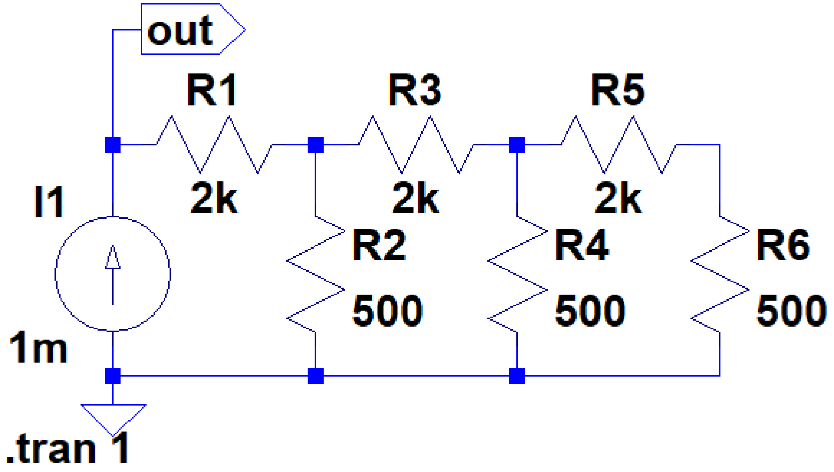
$$= \left(1 + \sqrt{2}\,\right)R$$

$$\approx 2.414R$$



Silver mean τ ladder

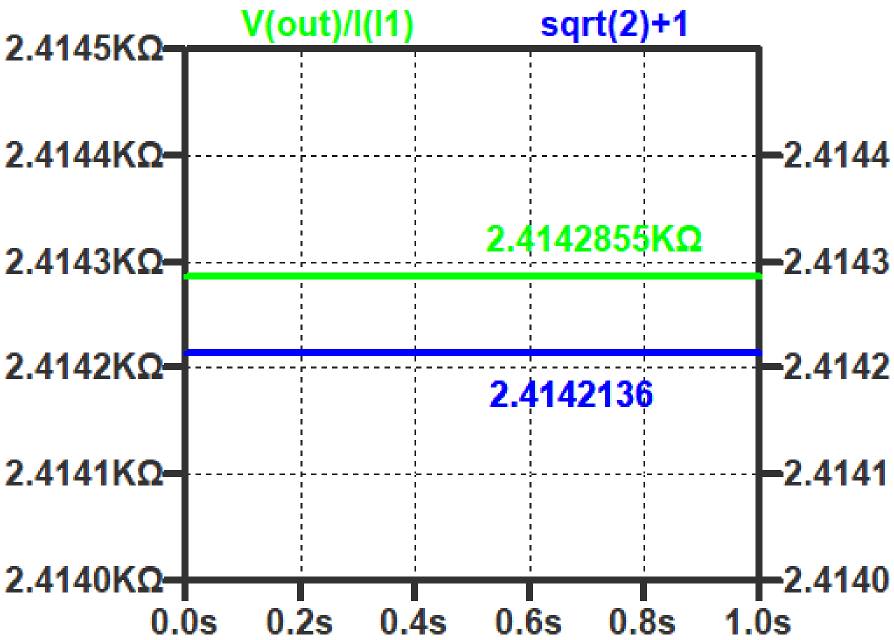
2R-0.5R Resistor Ladder (Simulation)



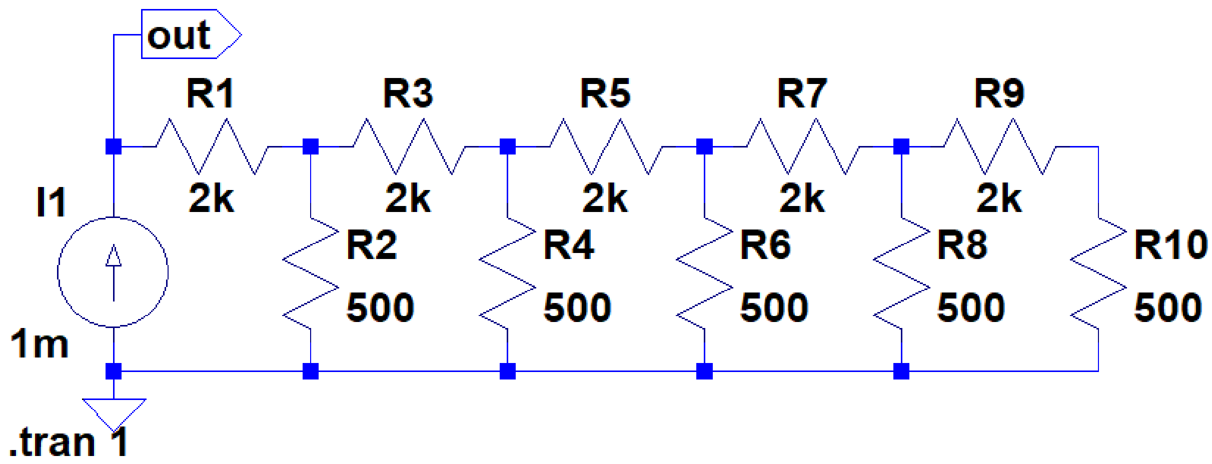
- Simulation conditions
  - R = 1 kΩ
  - Supply 1 mA to ladder, calculate Z_3 from $V(\text{out})$
- Result

Z_3 = 2.4142855 kΩ

$(1+\sqrt{2}$ = 2.4142135623$\cdots)$



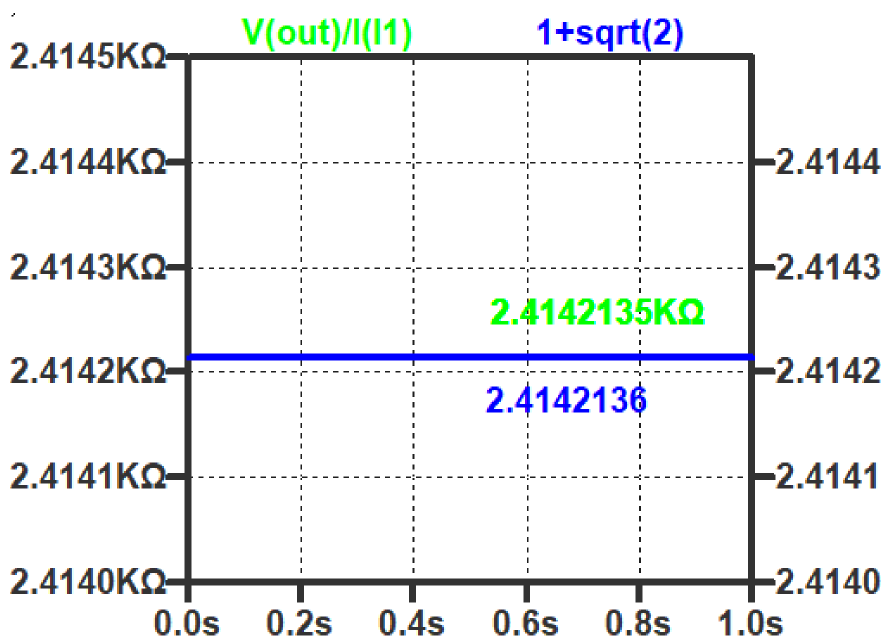
2R-0.5R resistor ladder



- Simulation conditions
  - R = 1 kΩ
  - Supply 1 mA to ladder, calculate Z_5 from $V(\text{out})$
- Result

$$Z_5 = 2.4142135\ \text{k}\Omega$$

$$(1+\sqrt{2} = 2.4142135623\cdots)$$

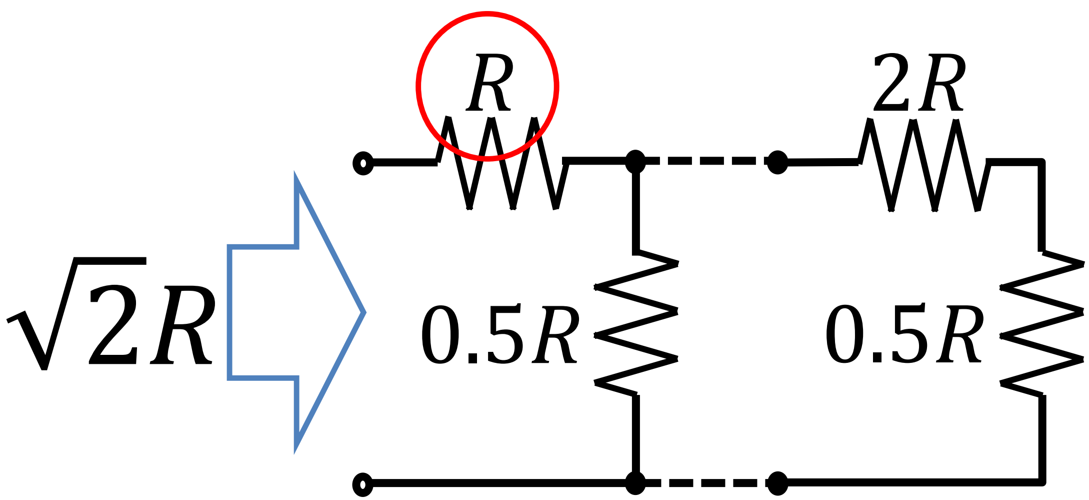


$\sqrt{2}$ Approximation Ladder

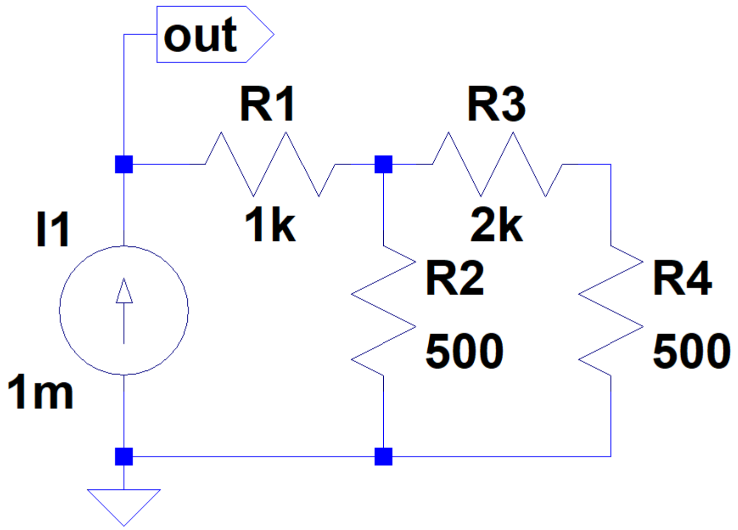
- $\sqrt{2} = \left(1+\sqrt{2}\right) - 1$
  $= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$

- $Z_{2R,0.5R} = \left(1+\sqrt{2}\,\right)R$
  $Z_{2R,0.5R} - R = \sqrt{2}R$

Replace the first 2R resistor of 2R-0.5R ladder with R.



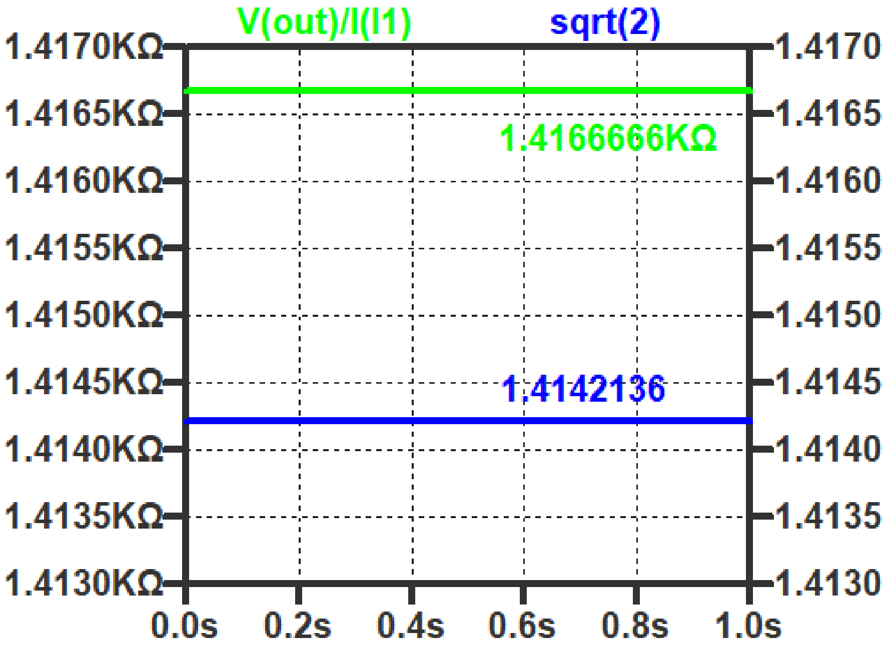
Verification of $\sqrt{2}$ Ladder, 2-Stage



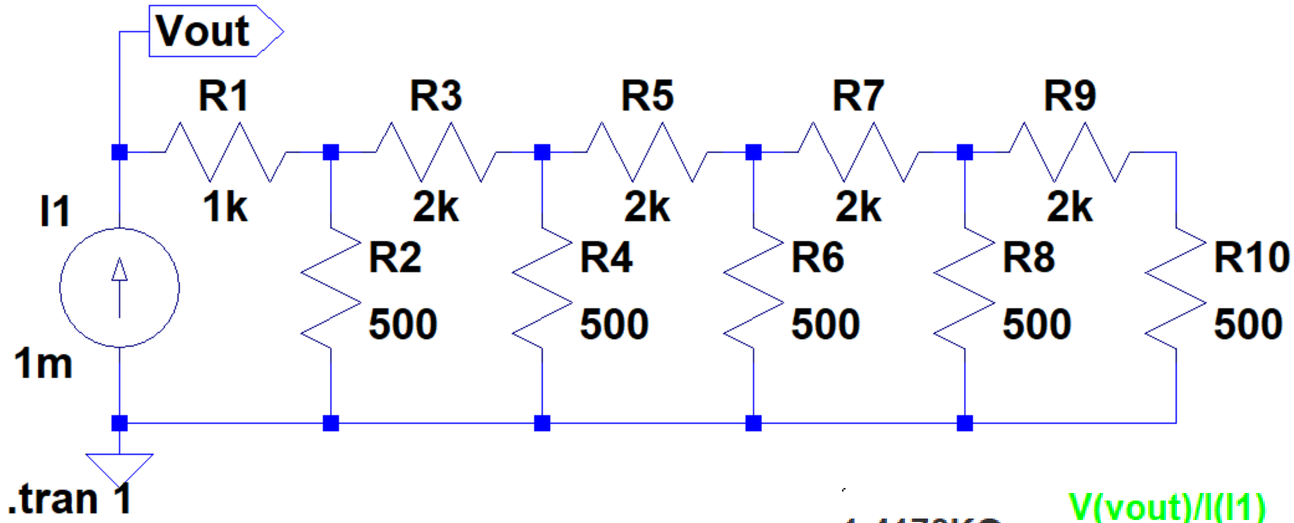
- Simulation conditions
  - R = 1 kΩ
  - Supply 1 mA to ladder, calculate Z_2 from $V(\mathrm{out})$
- Result

  Z_2 = 1.41666666 kΩ

  ($\sqrt{2}$ =1.41421356237309⋯)



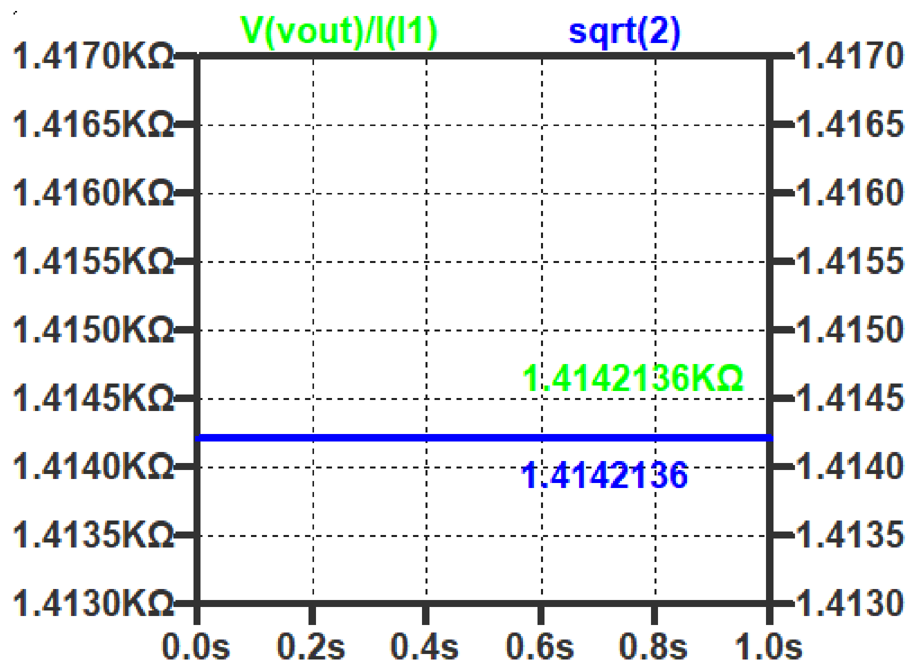
Verification of $\sqrt{2}$ Ladder, 5-Stage



- Simulation conditions
  - R = 1 kΩ
  - supply 1 mA to ladder, calculate Z_5 from $V(\text{out})$
- Result

  Z_5 = 1.4142136 kΩ

  ($\sqrt{2}$ =1.41421356237309⋯)



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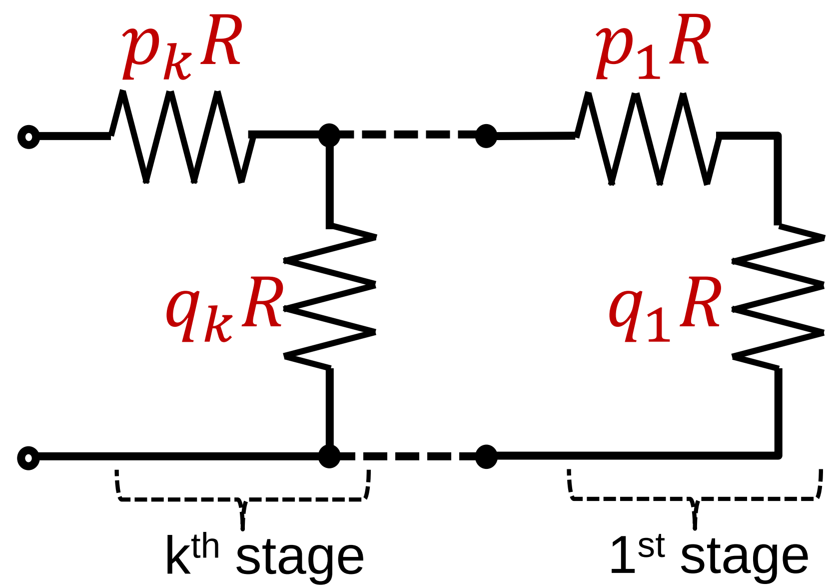
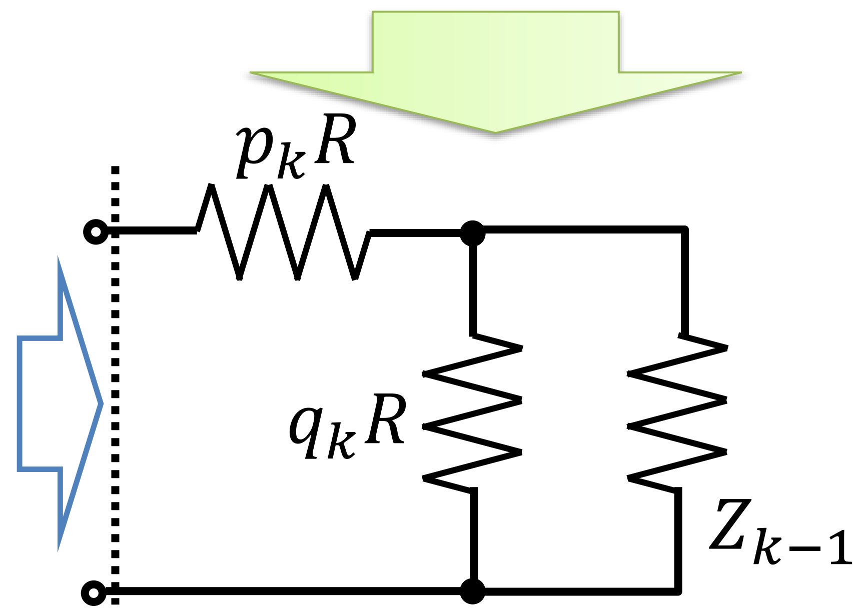
R-ladder with Different Resistance Values

Registance value of k-th stage
→ weighting by p_k and q_k



$$Z_k = p_k R + \frac{q_k R \cdot Z_{k-1}}{q_k R + Z_{k-1}}$$

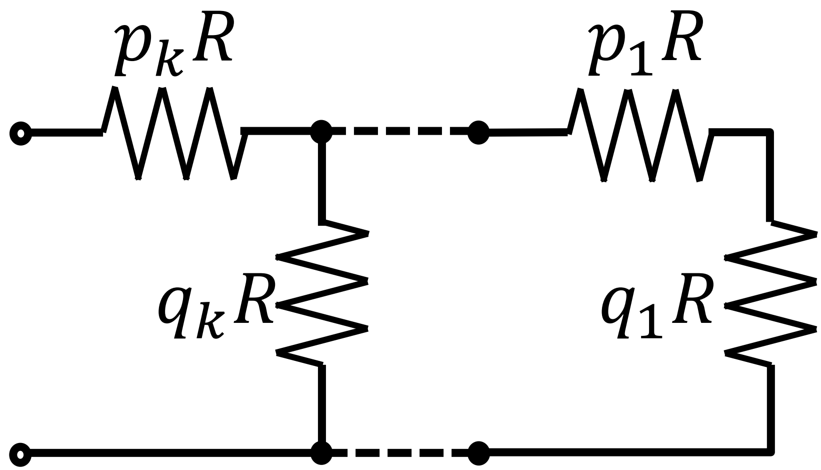


R-ladder with Different Resistance Value

$$Z_k = p_k R + \frac{q_k R \cdot Z_{k-1}}{q_k R + Z_{k-1}}$$

$$= R\left(p_k + \frac{1}{\frac{1}{q_k} + \frac{R}{Z_{k-1}}}\right)$$

$$= R\left(p_k + \cfrac{1}{\cfrac{1}{q_k} + \cfrac{1}{p_{k-1} + \cfrac{1}{\cfrac{1}{q_{k-1}} + \cfrac{1}{\ddots}}}}\right)$$



Adjust p_k and q_k according to continued fraction of specified number

⇨ Resistance ratio to R is specified number

Napier's Constant

- Irrational number
- Denoted by e
- Natural logarithm
- Continued fraction
  → regularity

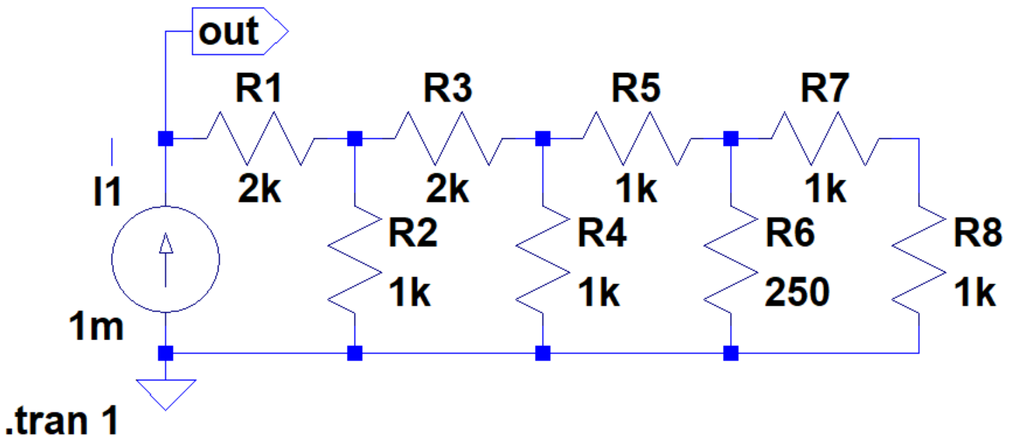
$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}$$

$$= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \cdots]$$

p_k → odd-numbered terms of integer part
2, 2, 1, 1, 6, …

q_k → reciprocals of even-numbered terms
1, 1, 1/4, 1, 1 , …

e Approximation Ladder, 4-Stage

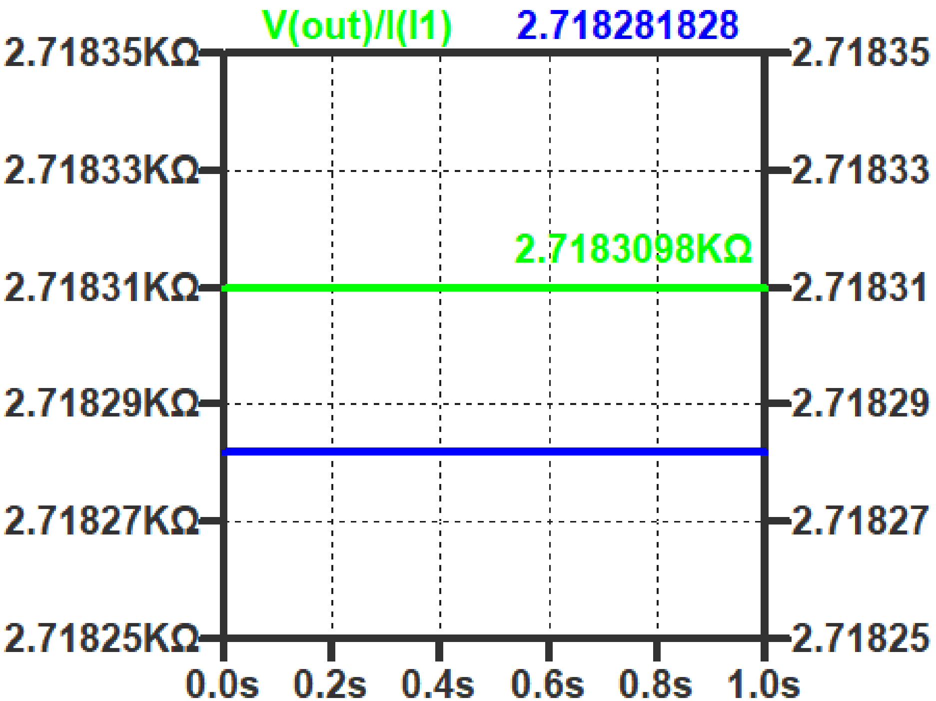


$$e \approx [2; 1, 2, 1, 1, 4, 1, 1]$$

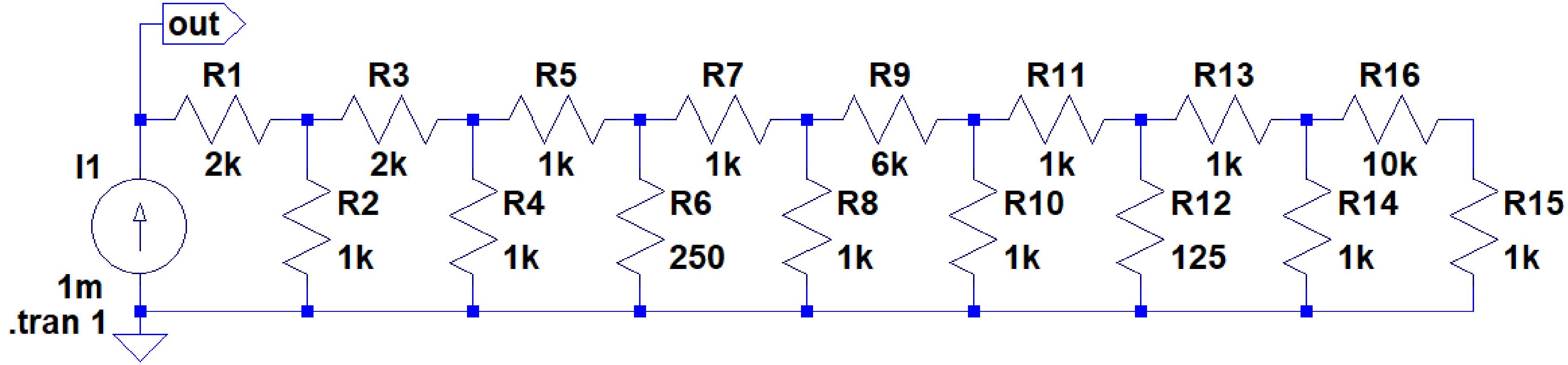
- Simulation condition
  - R = 1 kΩ
  - supply 1 mA to ladder, calculate Z_4 from $V(\text{out})$
- Result

  2.7183098 kΩ

  $(\mathrm{e} =$ 2.718281828459536 $\cdots)$



e Approximation Ladder, 8-Stage

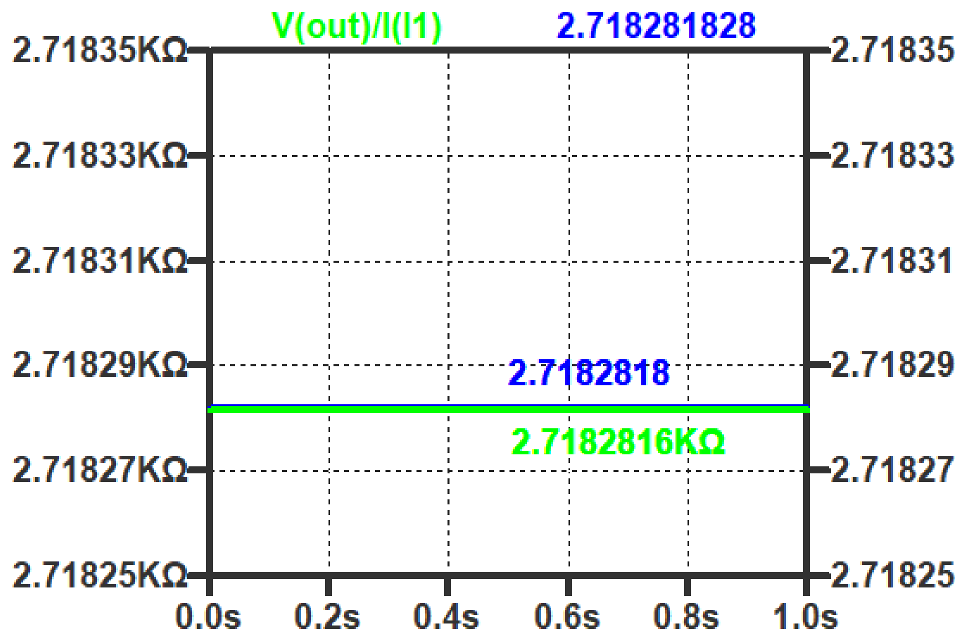


$$e \approx [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1]$$

- Simulation condition
  - $R = 1$ kΩ
  - supply 1 mA to ladder, calculate Z_8 from $V(\mathrm{out})$
- Result

  2.7182816 kΩ

  ($\mathrm{e} =$ 2.718281828459536 $\cdots$)



π Approximation Ladder

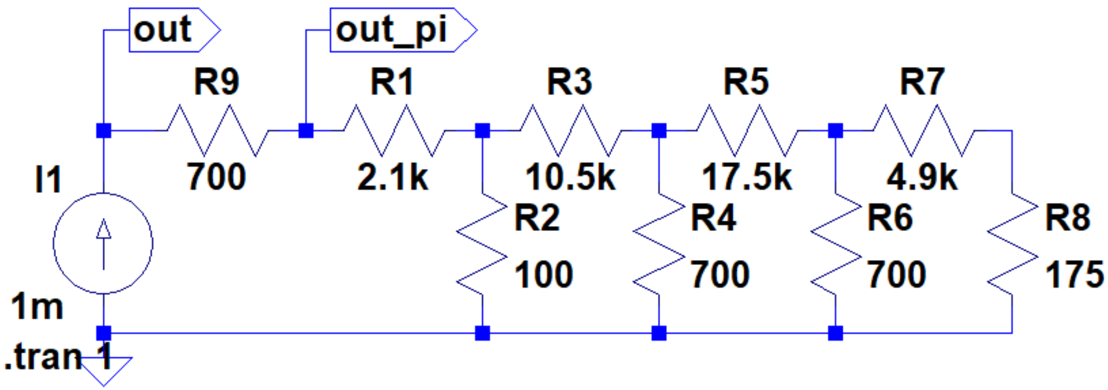
- Irrational number
- Ratio of a circle's circumference to diameter
- Continued fraction
  $\rightarrow$ no regularity

$$\pi \approx 3.14159$$
$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$
$$= [3; 7, 15, 1, 25, 1, 7, 4]$$

p_k $\rightarrow$ odd-numbered terms of integer part
3, 15, 25, 7

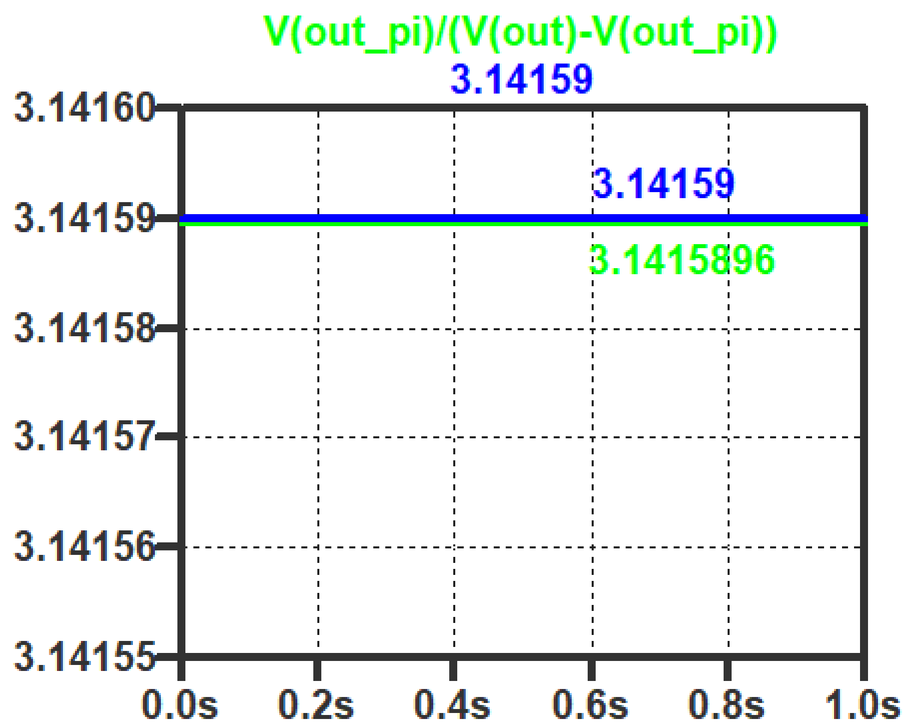
q_k $\rightarrow$ reciprocals of even-numbered terms
1/7, 1, 1, 1/4

π Approximation Ladder



- Simulation conditions
  - R = 700 Ω
  - Supply 1 mA to ladder and R, calculate resistance ratio to R from the ratio of voltages
- Result

Ratio to R : 3.1415896
(Design value : 3.14159)



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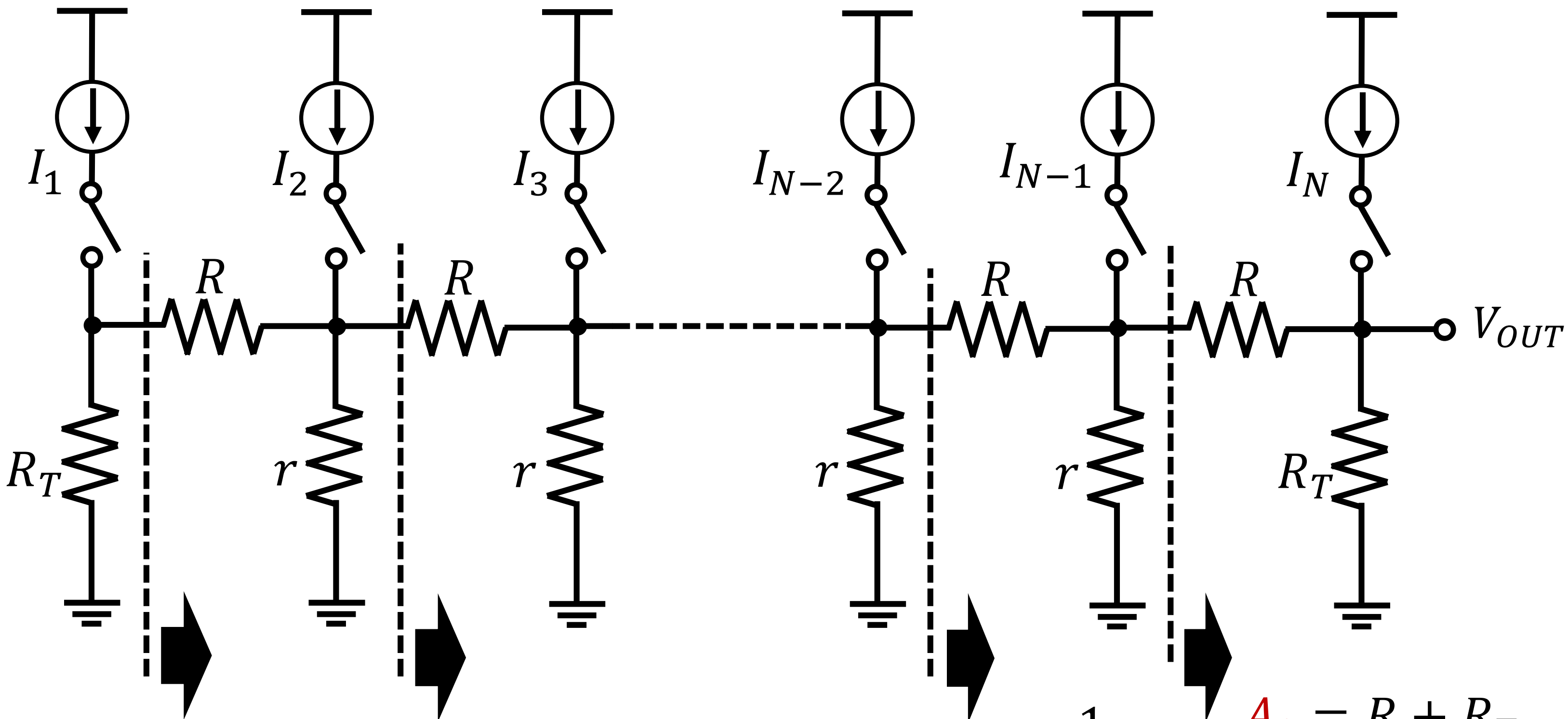
Resistor Network Digital-to-Analog Converters



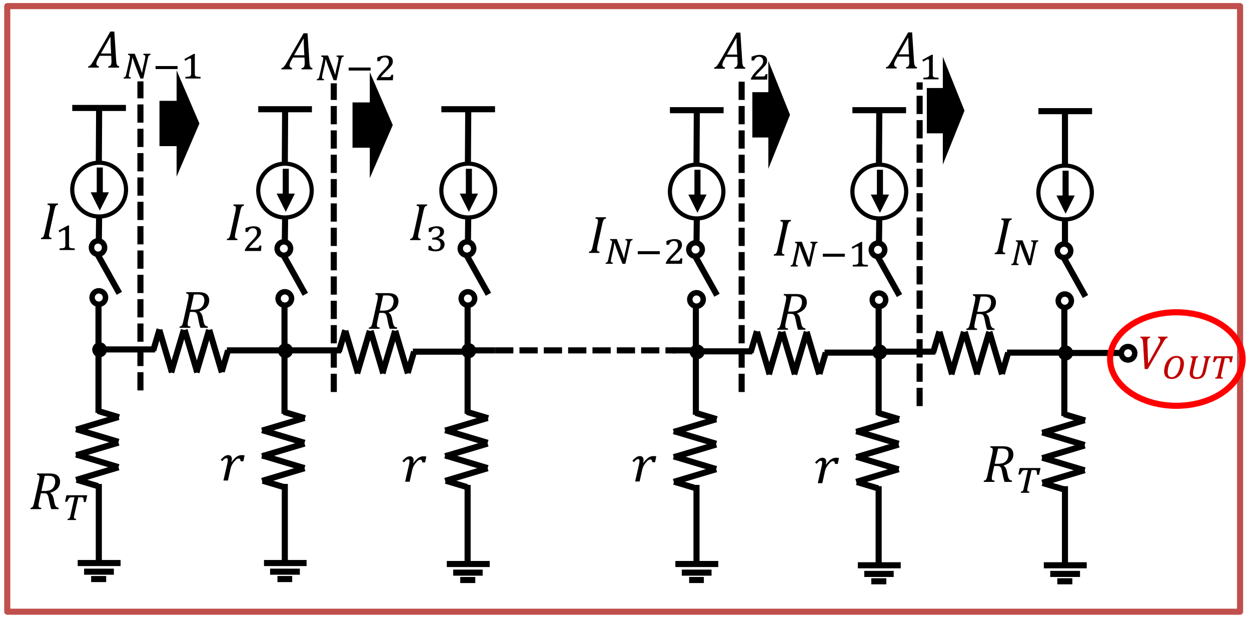
A_{N-1} A_{N-2}

$A_1 = R + R_T$

$$A_2 = R + \frac{1}{1/r + 1/A_1}$$

$$A_n = R + \frac{1}{1/r + 1/A_{n-1}}$$

Resistor Network Digital-to-Analog Converters



$$V_{OUT}(I_1, I_2, \cdots, I_N, R, r, R_T)$$
$$= R_T$$
$$\cdot \left(I_N \cdot \frac{A_{N-1}}{A_{N-1} + R_T} + I_{N-1} \cdot \frac{r||A_{N-2}}{r||A_{N-2} + A_1} + \frac{r}{A_1 + r} \right.$$
$$\cdot \left(I_{N-2} \cdot \frac{r||A_{N-3}}{r||A_{N-3} + A_2} + \frac{r}{A_2 + r} \right.$$

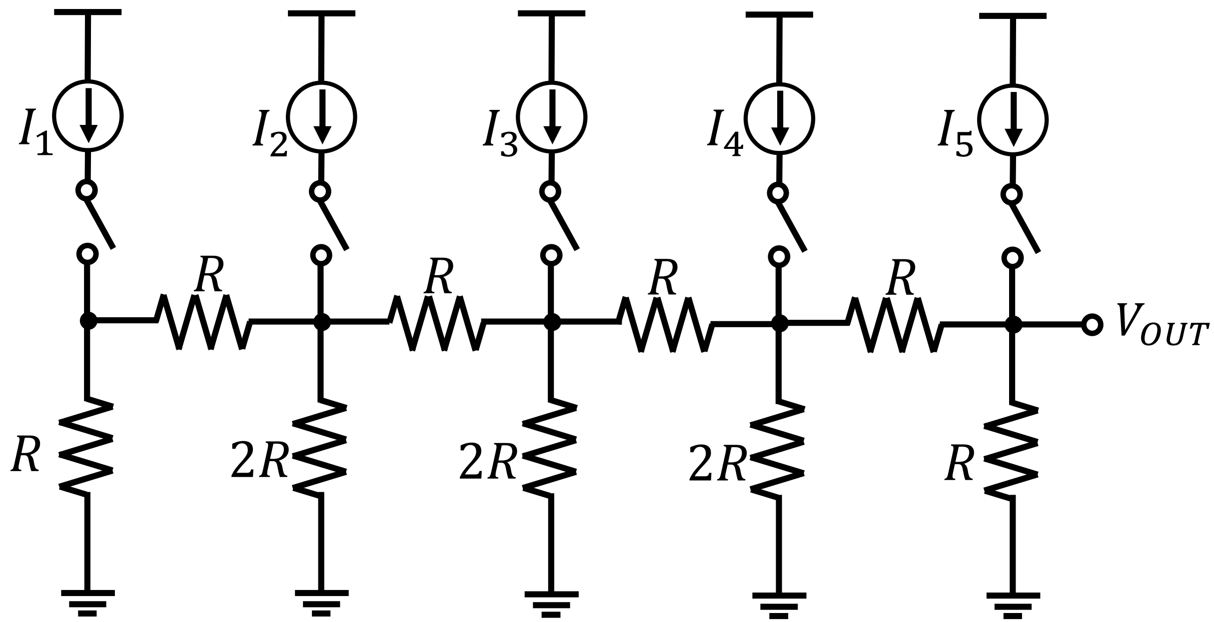
5-bit R-2R DAC

- R-2R DAC

$$R \to R$$
$$r \to 2R$$
$$R_T \to R$$

For all n,

$$A_n = 2R$$



$$V_{OUT}(I_1, I_2, I_3, I_4, I_5, R)$$

$$= R\left(I_5 \cdot \frac{2}{3} + I_4 \cdot \frac{1}{3} + \frac{1}{2} \cdot \left(I_3 \cdot \frac{1}{3} + \frac{1}{2} \cdot \left(I_2 \cdot \frac{1}{3} + \frac{1}{2} \cdot \left(I_1 \cdot \frac{1}{3}\right)\right)\right)\right)$$

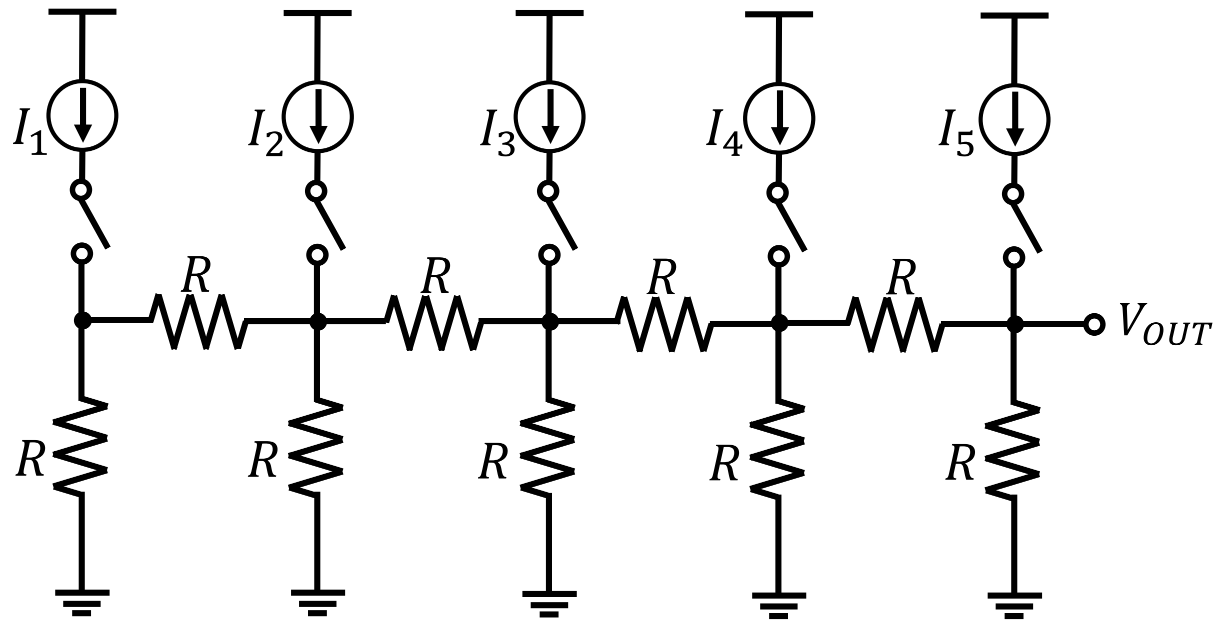
$$= \frac{1}{3} R\left(2I_5 + I_4 + \frac{1}{2} I_3 + \frac{1}{4} I_2 + \frac{1}{8} I_1\right)$$

⇦ Currents I weighted in binary

Examples of Resistor network DAC

- R-R network DAC

$$R \to R$$
$$r \to R$$
$$R_T \to R$$



$$V_{OUT}(I_1, I_2, I_3, I_4, I_5, R)$$
$$= R\left(I_5 \cdot \frac{34}{55} + I_4 \cdot \frac{13}{55} + \frac{1}{3} \cdot \left(I_3 \cdot \frac{3}{11} + \frac{3}{8} \cdot \left(I_2 \cdot \frac{16}{55} + \frac{8}{21} \cdot \left(I_1 \cdot \frac{21}{55}\right)\right)\right)\right)$$
$$= \frac{1}{55} R(34I_5 + 13I_4 + 5I_3 + 2I_2 + I_1)$$

When $I_1 \sim I_5 = (I, -\boldsymbol{I}, 0)$

⇨ Operate as a DAC

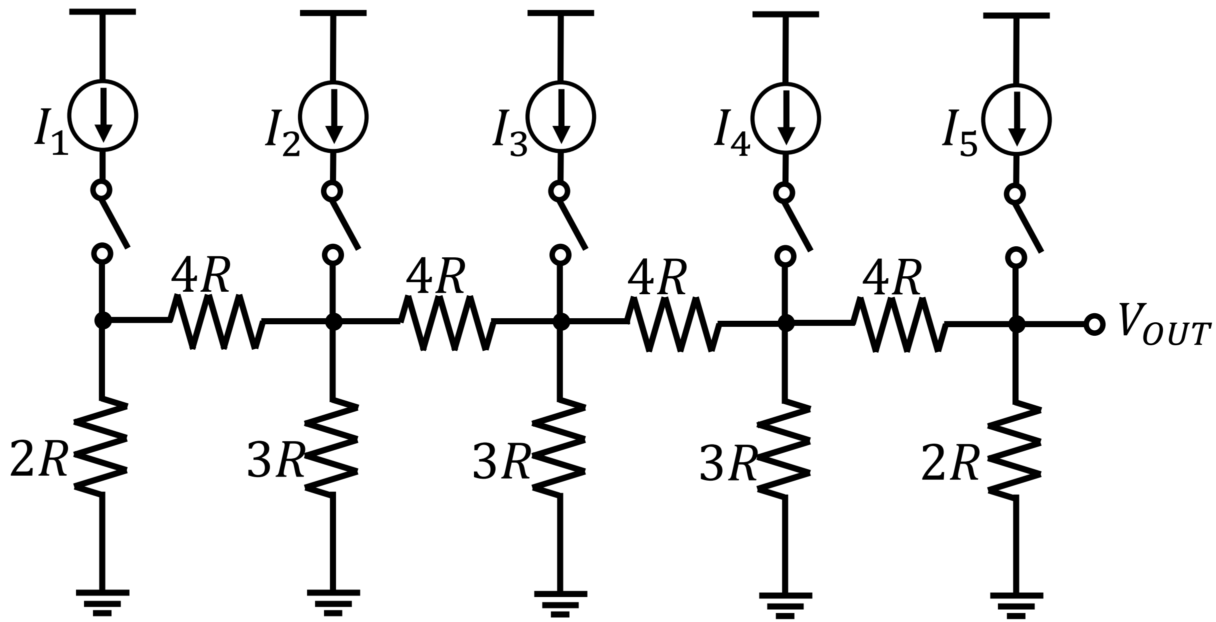
Examples of Resistor network DAC

- Weighted in "ternary"

$$R \to 4R$$
$$r \to 3R$$
$$R_T \to 2R$$

For all n,

$$A_n = 6R$$



$$V_{OUT}(I_1, I_2, I_3, I_4, I_5, R)$$
$$= 2R\left(I_5 \cdot \frac{6}{8} + I_4 \cdot \frac{2}{8} + \frac{1}{3} \cdot \left(I_3 \cdot \frac{2}{8} + \frac{1}{3} \cdot \left(I_2 \cdot \frac{2}{8} + \frac{1}{3} \cdot \left(I_1 \cdot \frac{2}{8}\right)\right)\right)\right)$$
$$= \frac{1}{2}R\left(3I_5 + I_4 + \frac{1}{3}I_3 + \frac{1}{9}I_2 + \frac{1}{27}I_1\right).$$

When $I_1 \sim I_5 = (I, -\boldsymbol{I}, 0)$

⇨ Operate as a DAC

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Conclusion

- Clarified
  - R-r ladder network ⇔ some irrational numbers
  - By using continued fraction of specified number, equivalent resistor (resistance ratio to R is specified number)
  - Approximation accuracy $\rightarrow$ better, as the number of resistors larger
- Resistor network DAC
  - Generalized DAC using resistor ladder
  - New idea of Non-binary DACs

Q&A

- Q. You showed a ternary DAC on slide 32. Can you design a “Quaternary” DAC?
  A. Probably, we can. But I suppose a “Quaternary” DAC needs more current sources than ternary DAC.

- Q. Can you design the $\sqrt{2}$ weighted DAC?
  A. I’m not sure. Probably we can by using a $1+\sqrt{2}$ ladder or other configuration of ladder.