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Floating-point Square Root Calculation Arithmetic Based on Taylor-Series Expansion and Region Division

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Outline

- Research Background and Objective
- Taylor-Series Expansion
- Proposed Algorithm
- Simulation Verification
- Hardware Implementation Tradeoff
- Conclusion

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Research Background

◆High-speed high-precision floating-point arithmetic

→ Scientific computing
Embedded systems
Mobile applications

◆Square Root calculation ⇒ Very tough !

Multiple usage of basic arithmetic operations

- ➢ Addition / Subtraction → Relatively easy
- ➢ Multiplication → Modestly complicated

Research Objective

- Floating-point square root calculation
  - Simple circuit
  - High-speed
- Application of Taylor-series expansion with divide-and-conquer of mantissa region
- Clarification of
  - calculation algorithm
  - design tradeoff among accuracy, number of operations and LUT size.

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Taylor Series Expansion

Re-write a smooth function
as an infinite sum of polynomial terms.

Function $f(x)$ for a point $x = a$ is given by :

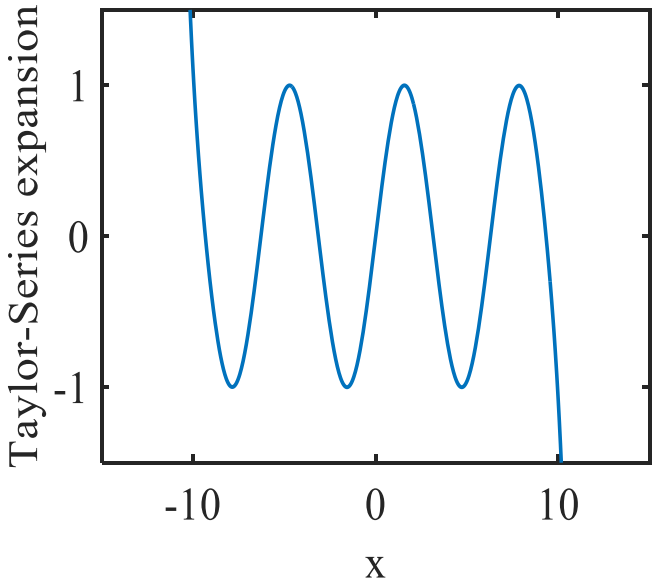
$$f(x) = f(a) + f'(a)(x-a) + \frac{(f)''(a)}{2!}(x-a)^2 + \cdots + \frac{(f)^n(a)}{n!}(x-a)^n + \cdots$$

Convergence range $\alpha < x < \beta$

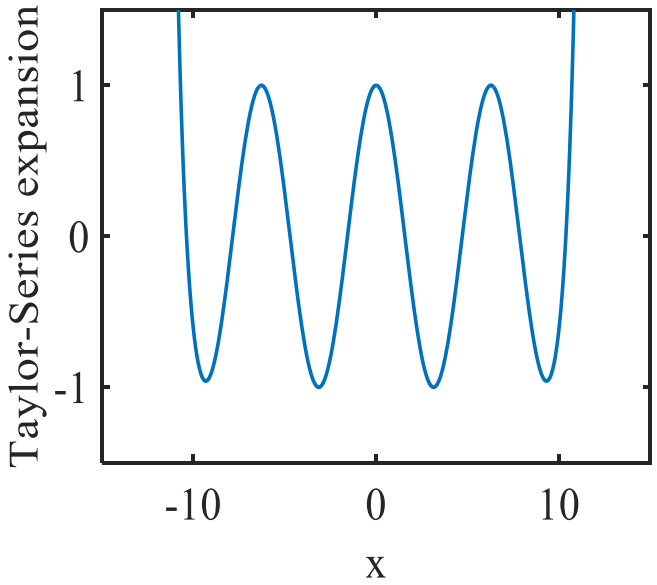
Taylor-Series of sin(x), cos(x)

Taylor-series expansion with 20 terms.

for center value : a=0



$\sin(x)$



$\cos(x)$

Convergence range : $-\infty < x < +\infty$

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Normalized Floating-Point Representation

Floating-point representation in binary :

Mantissa : M　　(1 ≦ M < 2)　　Exponent : E

$$M \times 2^{E}$$

Decimal point

$$1.abcdef\cdots \times 2^{E}$$

Mantissa　　Exponent

$a, b, c, d, e, f, \cdots$: 0 or 1

Square Root Calculation

Floating-point binary : $S = M \times 2^E$

Square root calculation of S

$$\sqrt{S} = \sqrt{M} \times \sqrt{2^E}$$

E: even $\rightarrow$ Let $E=2k$

$$\sqrt{S} = \sqrt{M} \times 2^k$$

E: odd $\rightarrow$ Let $E=2k+1$

$$\sqrt{S} = \sqrt{M} \times \sqrt{2} \times 2^k$$

$\sqrt{M}$ calculation by Taylor-series expansion for a specified accuracy.

Usage of Taylor Expansion

Calculate square root of mantissa : $\sqrt{M}$ ($1 \leq M < 2$)

$x = M$

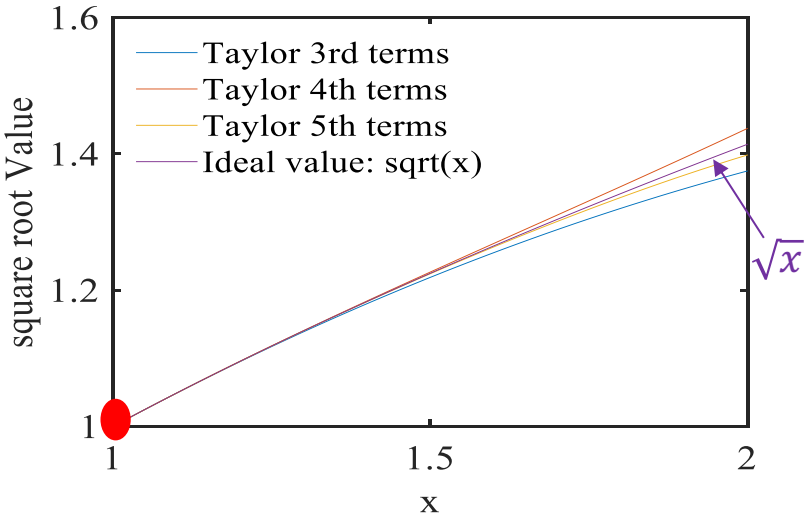
$f(x) = \sqrt{x}$ by Taylor expansion at $x = a$ $(1 \leq a < 2)$

$f(x) = \sqrt{x}$ Taylor expansion at $x = a$:

$$f(x) = \sqrt{a} \times \left\{ 1 + \frac{x - a}{2} - \frac{(x - a)^2}{8 \times a} + \frac{(x - a)^3}{16 \times a^2} - \frac{5 \times (x - a)^4}{128 \times a^3} + \frac{7 \times (x - a)^5}{256 \times a^4} - \cdots \right\}$$

Coefficient values are stored in LUT in advance.

Mantissa Region Division



Taylor series expansion of $\sqrt{x}$ at center value a = 1

Divide and Conquer of Mantissa Region

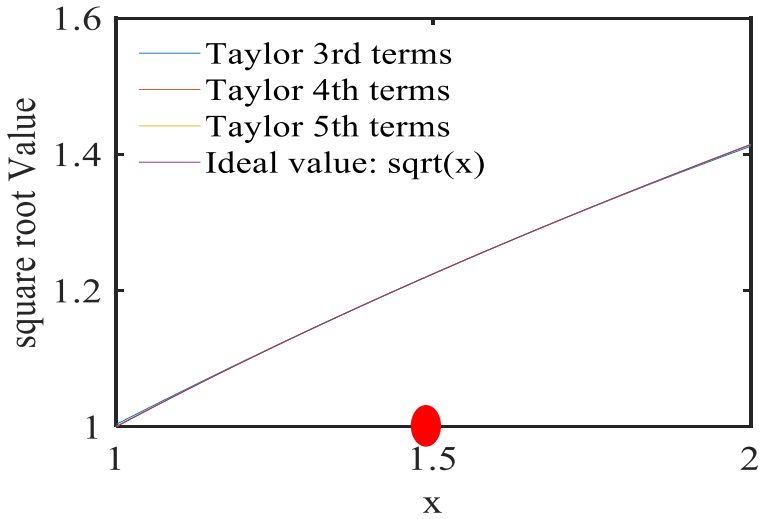
1 region :

a = 1.5 $1 \leq x < 2$

2 regions :

a = 1.25 $1 \leq x < 1.5$

a = 1.75 $1.5 \leq x < 2$



Taylor series expansion of $\sqrt{x}$ at center value a = 1.5

4 regions :

a = 1.125 $1 \leq x < 1.25$

a = 1.375 $1.25 \leq x < 1.5$

a = 1.625 $1.5 \leq x < 1.75$

a = 1.875 $1.75 \leq x < 2$

⋮

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Definition of Accuracy

Ex : $\frac{1}{2^{16}}$ accuracy

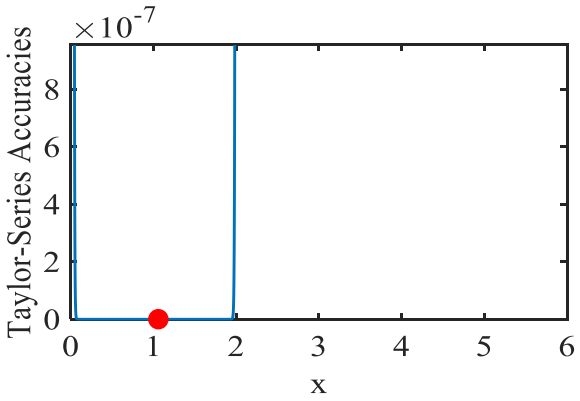
$$max \left| \frac{f(x) - t(n,x)}{f(x)} \right| \leq \frac{1}{2^{16}}$$

$f(x)$: Original function (Ideal value)

$t(n,x)$: Taylor expansion of n terms

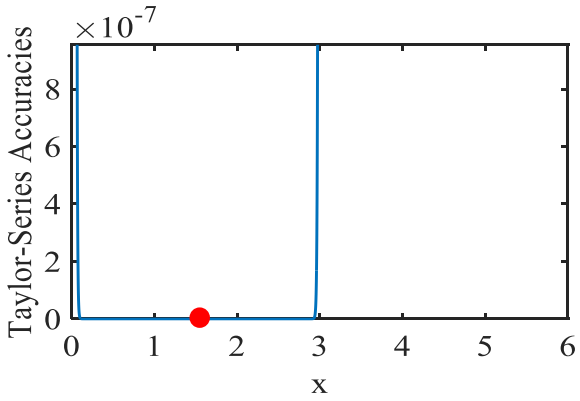
Graph of $\sqrt{x}$ Taylor Expansion

Number of Taylor expansion terms : 200



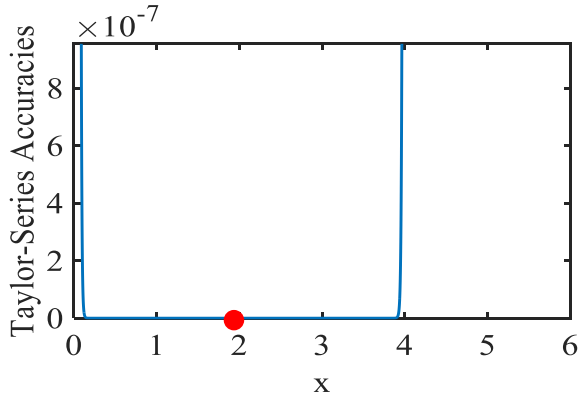
a = 1

Convergence range : 0 < *x* < 2



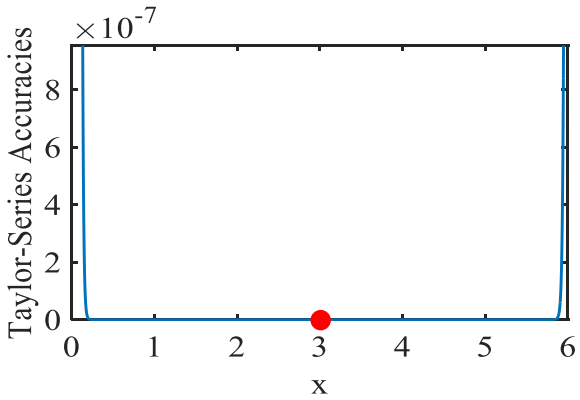
a = 1.5

Convergence range : 0 < *x* < 3



a = 2

Convergence range : 0 < *x* < 4



a = 3

Convergence range : 0 < *x* < 6

One-Region Case

Use Taylor series expansion equation :

$f(x) = \sqrt{x}$ $\quad (1 \leq x < 2)$

Mantissa represented by binary decimal point.

Specified accuracy

Taylor-series expansion \ precision		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) $M =$ 1.xxxxxx… $1 \leq M < 2$	$a = 1.5$	3	7	9	12	15

Taylor series expansion at center value $a = 1.5$

Number of Taylor expansion terms to meet specified accuracy.

Two-Region Case

Use Taylor series expansion equation :

$f(x) = \sqrt{x}$ $\quad (1 \leq x < 2)$

We check value (0 or 1)
of the first decimal place of Mantissa.

Taylor-series expansion / precision		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) $M_D = 1.0xxxxx\cdots$ $1 \leq M_D < 1.5$	$a = 1.25$	3	5	7	8	11
(ii) $M_D = 1.1xxxxx\cdots$ $1.5 \leq M_D < 2$	$a = 1.75$	2	5	6	7	10

Four-Region Case

Use Taylor series expansion equation :

$f(x) = \sqrt{x} \qquad (1 \leq x < 2)$

We check values (00, 01, 10 or 11)
of the first two decimal places of Mantissa.

Taylor-series expansion \ precision		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) M = 1.00xxxx… $1 \leq M_D < 1.25$	a=1.125	2	4	5	6	9
(ii) M = 1.01xxxx… $1.25 \leq M_D < 1.5$	a=1.375	2	4	5	6	8
(iii) M = 1.10xxxx… $1.5 \leq M_D < 1.75$	a=1.625	2	4	5	6	8
(iv) M = 1.11xxxx… $1.75 \leq M_D < 2$	a=1.875	2	4	5	5	7

Eight-Region Case

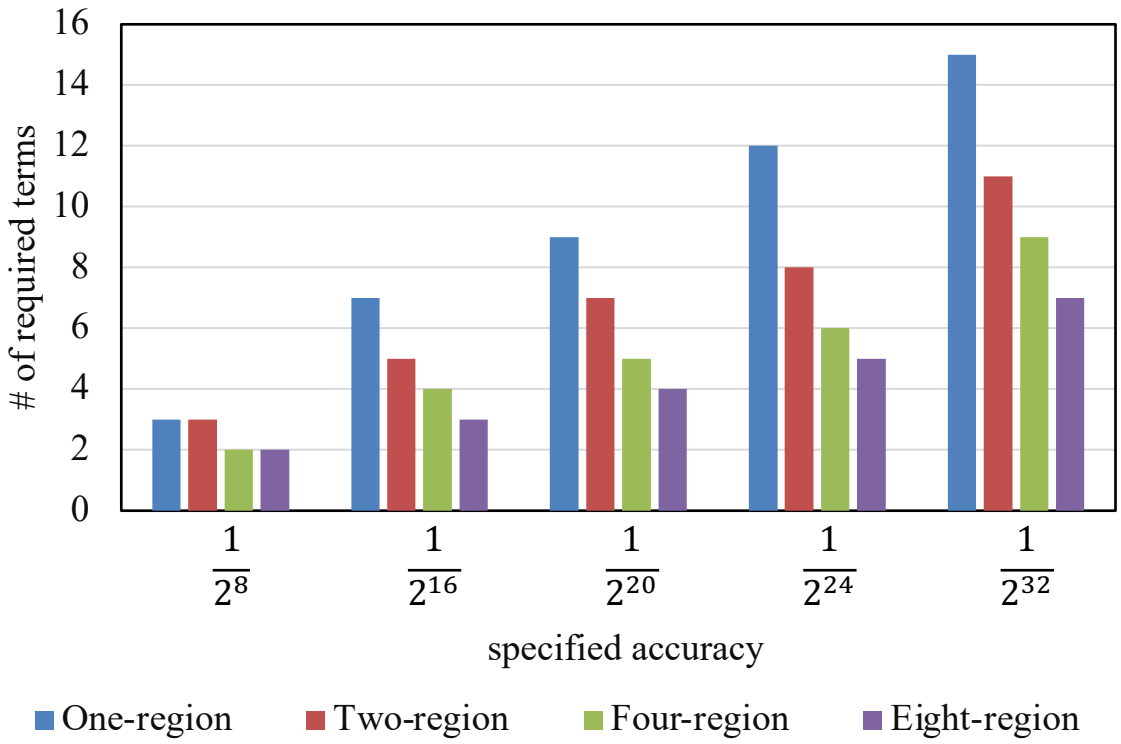
Check the values (000, 001,…, 111)
of the first three decimal places of Mantissa.

Taylor-series expansion \ precision		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) M = 1.000xxxx… $1 \le M_D < 1.125$	a = 1.0625	2	3	4	5	7
(ii) M = 1.001xxxx… $1.125 \le M_D < 1.25$	a = 1.1875	2	3	4	5	7
(iii) M = 1.010xxxx… $1.25 \le M_D < 1.375$	a=1.3125	2	3	4	5	7
(iv) M = 1.011xxxx… $1.375 \le M_D < 1.5$	a=1.4375	2	3	4	5	6
(v) M = 1.100xxxx… $1.5 \le M_D < 1.625$	a=1.5625	2	3	4	5	6
(vi) M = 1.101xxxx… $1.625 \le M_D < 1.75$	a=1.6875	2	3	4	5	6
(vii) M = 1.110xxxx… $1.75 \le M_D < 1.875$	a=1.8125	2	3	4	4	6
(viii) M = 1.111xxxx… $1.875 \le M_D < 2$	a=1.9375	2	3	4	4	6

Comparison of Number of Required Terms

Comparison of
required number of terms
for different number of region division

Taylor-series expansion \ precision		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) $M_D = 1.0xxxxx\cdots$ $1 \le M_D < 1.5$	$a = 1.25$	3	5	7	8	11
(ii) $M_D = 1.1xxxxx\cdots$ $1.5 \le M_D < 2$	$a = 1.75$	2	5	6	7	10



Number of divided regions
becomes larger

Number of terms reduced

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Calculation Complexity

- In case of Taylor expansion 5 terms :

$$f_5(x) = \sqrt{a} \times \left\{1 + \frac{x-a}{2} - \frac{(x-a)^2}{8\times a} + \frac{(x-a)^3}{16\times a^2} - \frac{5\times(x-a)^4}{128\times a^3}\right\}$$
$$= \alpha_0[2 + (x-a)] - \alpha_2(x-a)^2 + \alpha_3(x-a)^3 - \alpha_4(x-a)^4$$

Here, $\alpha_0 = \frac{\sqrt{a}}{2}, \alpha_2 = \frac{\sqrt{a}}{8\times a}, \alpha_3 = \frac{\sqrt{a}}{16\times a^2}, \alpha_4 = \frac{5\sqrt{a}}{128\times a^3}$.

***E* (exponent) Even case.**

$$g_5(x) = \sqrt{2} \times f_5(x)$$
$$= \beta_0[2 + (x-a)] - \beta_2(x-a)^2 + \beta_3(x-a)^3 - \beta_4(x-a)^4$$

Here, $\beta_0 = \sqrt{2}\alpha_0,\ \beta_2 = \sqrt{2}\alpha_2,\ \beta_3 = \sqrt{2}\alpha_3,\ \beta_4 = \sqrt{2}\alpha_4$.

***E* (exponent) Odd case.**

$\alpha_0, \alpha_2, \alpha_3, \alpha_4, \beta_0, \beta_2, \beta_3, \beta_4$ values : Stored in LUT and read.

y = x−a Subtraction: 1 time

z = y^2 Multiplication: 1 time

$f_5(x) = \alpha_0(2 + y) - z(\alpha_2 - \alpha_3 y + \alpha_4 z)$

Multiplication: 4 times
Addition / Subtraction: 4 times

Total : Multiplication: 5 times
Addition / Subtraction: 5 times

Number of Operations

Number of terms versus number of operations in Taylor expansion

Taylor expansion of $f(x) = \sqrt{x}$ can be calculated with a small number of Mul/Add/Sub operations.

Terms of Taylor expansion	multiplication	Addition or subtraction
3	3	3
4	4	4
5	5	5
6	6	6
7	7	7
8	8	8

In case of 5 terms

LUT Size

$$f_5(x) = \sqrt{a}\times\left\{1+\frac{x-a}{2}-\frac{(x-a)^2}{8\times a}+\frac{(x-a)^3}{16\times a^2}-\frac{5\times(x-a)^4}{128\times a^3}\right\}$$
$$= \alpha_0[2+(x-a)]-\alpha_2(x-a)^2+\alpha_3(x-a)^3-\alpha_4(x-a)^4$$

E (exponent) Even case.

Here, $\alpha_0=\frac{\sqrt{a}}{2}, \alpha_2=\frac{\sqrt{a}}{8\times a}, \alpha_3=\frac{\sqrt{a}}{16\times a^2}, \alpha_4=\frac{5\sqrt{a}}{128\times a^3}$.

$$g_5(x) = \sqrt{2}\times f_5(x)$$
$$= \beta_0[2+(x-a)]-\beta_2(x-a)^2+\beta_3(x-a)^3-\beta_4(x-a)^4$$

E (exponent) Odd case.

Here, $\beta_0=\sqrt{2}\alpha_0,\ \beta_2=\sqrt{2}\alpha_2,\ \beta_3=\sqrt{2}\alpha_3,\ \beta_4=\sqrt{2}\alpha_4$.

$\alpha_0, \alpha_2, \alpha_3, \alpha_4, \beta_0, \beta_2, \beta_3, \beta_4$ values : Stored in LUT and read.

4 - region case → LUT size of 32 words

Address (M=1.*ab*⋯)	LUT data
00	$\alpha_0, \alpha_2, \alpha_3, \alpha_4,\ \beta_0, \beta_2, \beta_3, \beta_4$ for $a = 1.125$
01	$\alpha_0, \alpha_2, \alpha_3, \alpha_4,\ \beta_0, \beta_2, \beta_3, \beta_4$ for $a = 1.357$
10	$\alpha_0, \alpha_2, \alpha_3, \alpha_4,\ \beta_0, \beta_2, \beta_3, \beta_4$ for $a = 1.625$
11	$\alpha_0, \alpha_2, \alpha_3, \alpha_4,\ \beta_0, \beta_2, \beta_3, \beta_4$ for $a = 1.875$

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Conclusion

- Square root calculation of mantissa in binary floating format using Taylor expansion has been investigated.
- Proposed mantissa region division method can effectively control the number of Taylor series expansion terms.
- Number of divided regions becomes larger :
  - ➢Number of Taylor expansion terms ➡ smaller
  - ➢LUT size ➡ larger.

Divide-and-Conquer is effective.

Thank you for listening !

Comment

Form Prof. Robert Brennan

https://schulich.ucalgary.ca/contacts/robert-brennan

Comment:

1. It would be more efficient to calculate 1/sqrt(x) using fixed point iteration followed by multiplying by x. This would converge in 2 iterations.
2. you need 4 bit accuracy for convergence to 16 bits. A second order polynomial will give you that. Global convergence is actually excellent given a 4-bit accurate input. Derivative is well behaved.