ATS Doctoral Thesis Award

Virtual Event Hosted by Japan, Nov. 22-24, 2021

Semi-Final of 2022 TTTC's E. J. McCluskey Doctoral Thesis Award

ΔΣ ADC Linearity Testing Technology and Floating-Point Arithmetic Algorithms with Taylor-Series Expansion

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Kobayashi Laboratory

Outline

1. **Research Background**
2. **ΔΣ ADC Linearity Testing Technology**
   - **ΔΣ ADC Testing challenge and Linearity**
   - **Proposed linearity test method**
   - **Simulation configuration and results**
3. **Floating-Point Arithmetic Algorithms withTaylor-Series Expansion**
   - **Taylor-Series Expansion and Proposed Algorithm**
   - **Simulation Verification**
   - **Hardware Implementation Consideration**

4.Conclusion

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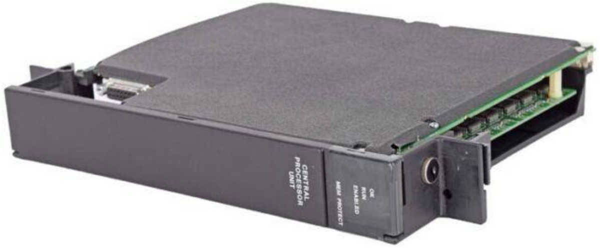
4. **Conclusion**

Research Background

- IoT devices are becoming important.
- Their high quality and low cost testing is required.
- High-speed, high-precision floating-point DSP in ATE systems.

Efficient algorithms are required.

ATE (Automatic Test Equipment)



Floating-point processor

Research Objective (1)

High resolution, **low speed** **ΔΣ ADC**

- ➢ Sensor interface
- ➢ Mass production test

Linearity test takes a **long time**.

→ Usually omitted.

High reliability requirements

- ✓ **Its linearity test in short time**
- ✓ **Develop its algorithm**

Research Objective (2)

- Floating-point **arithmetic** calculation
  - **Simple circuit**
  - **High-speed**

Divide difficulties.

René Descartes

- Application of Taylor-series expansion with **divide-and-conquer** of **mantissa region**
- Clarification of
  - Calculation algorithm
  - Design tradeoff among accuracy, number of operations and **LUT** size.

LUT : **Look-Up Table**

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ΔΣ ADC Testing Challenge

Sensor + amplifier + ΔΣADC + microcomputer

- **Complicated ADC output signal processing** ➡ **Reproduction circuit**
- **Low speed sampling, High resolution** ➡ **Long test time**
- **High linearity analog input signal** ➡ **High precision signal generator**

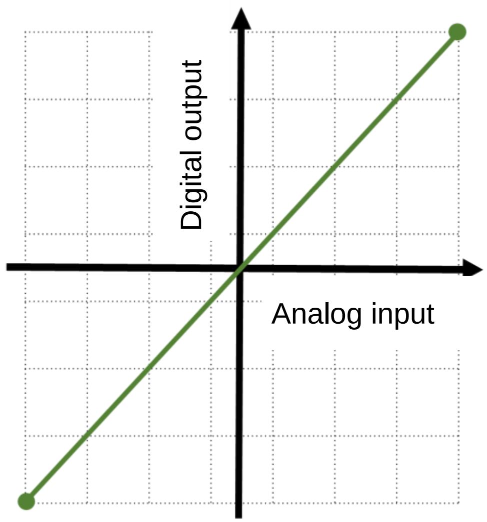
1 US dollar chip

⬇

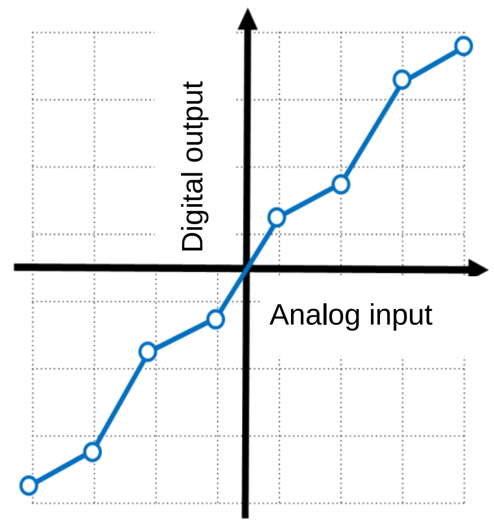
Test time should be less than 1 second

Linearity of ΔΣ ADC

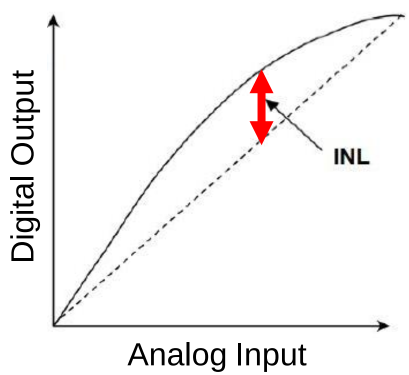
Ideal characteristic **(linear)**



Actual (**nonlinear**)



Circuit imperfection, variation → **Nonlinear**



Large INL →

- **Missing codes**
- **Lack of monotonicity**

INL: Integral Non-Linearity

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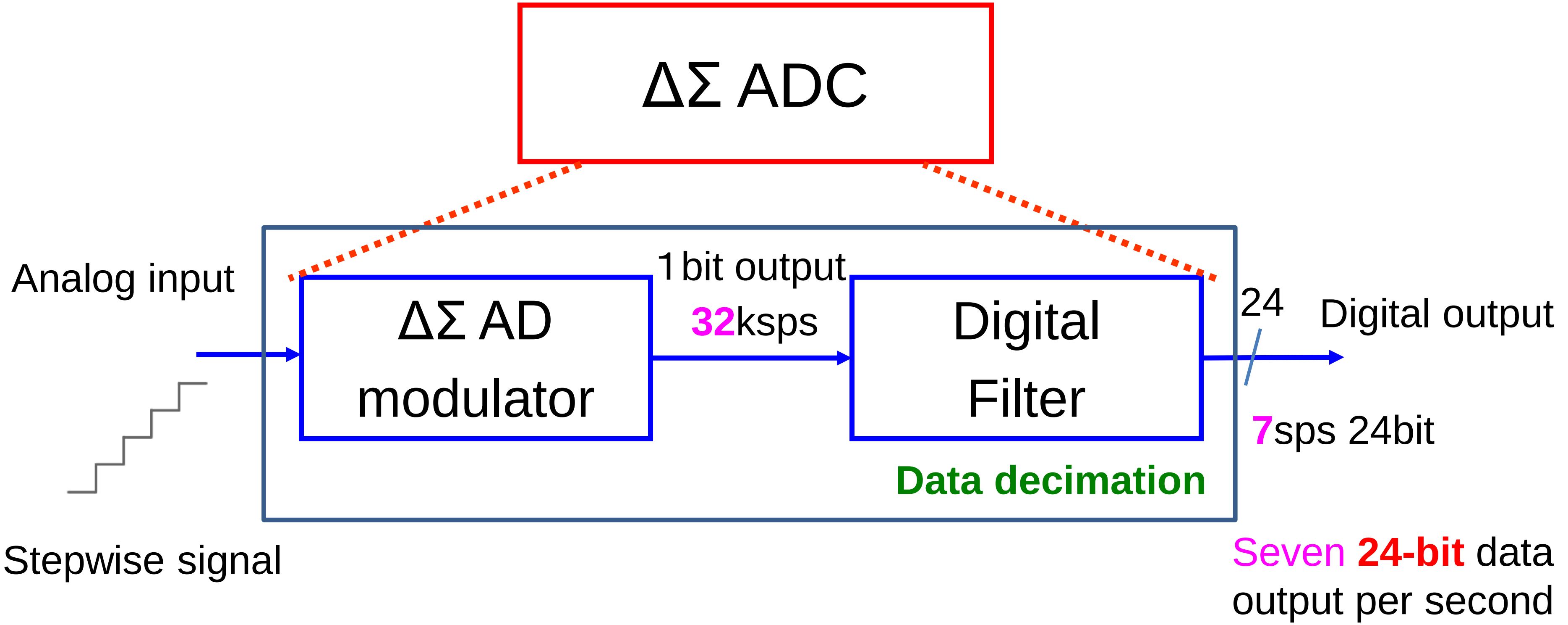
Problem of Direct Linearity Test

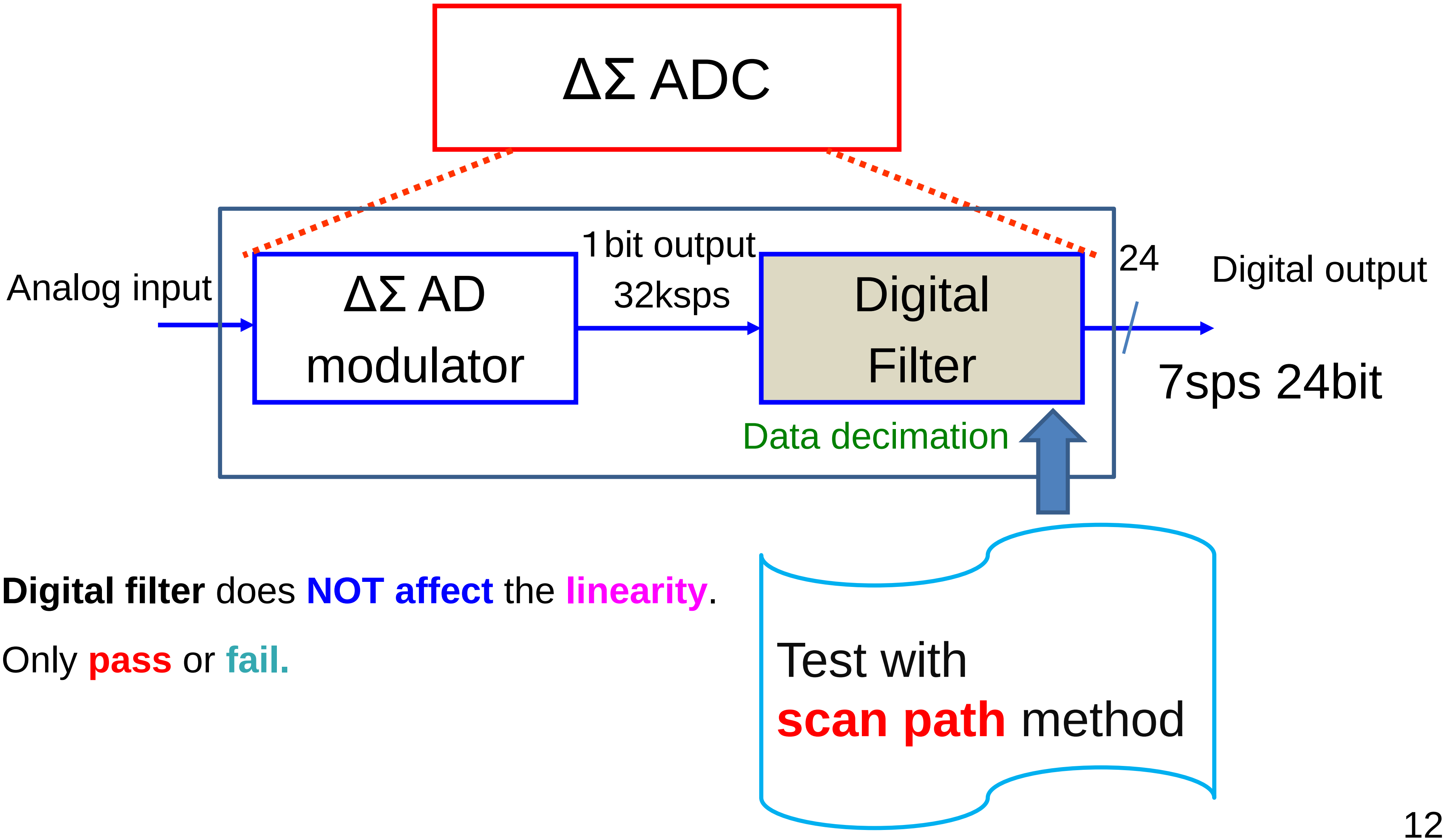


Linearity test time:

4 point per code

(1/7) x 2^{24} x 4 **seconds** = **104 day** ➡ **Totally unrealistic**

Proposed: Digital Filter Test

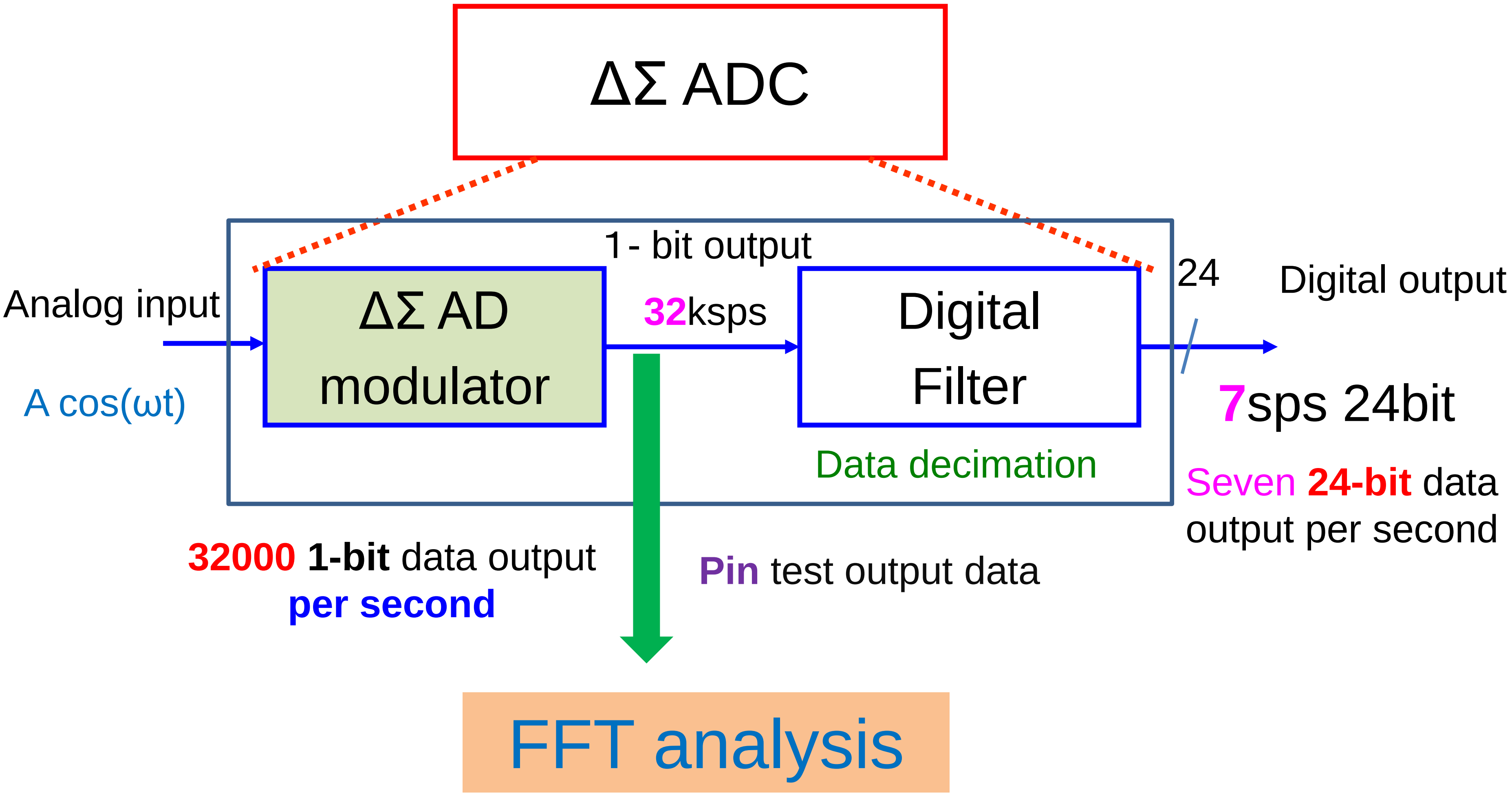


Digital filter does **NOT affect** the **linearity**.

Only **pass** or **fail.**

Proposed: ΔΣ AD Modulator Test

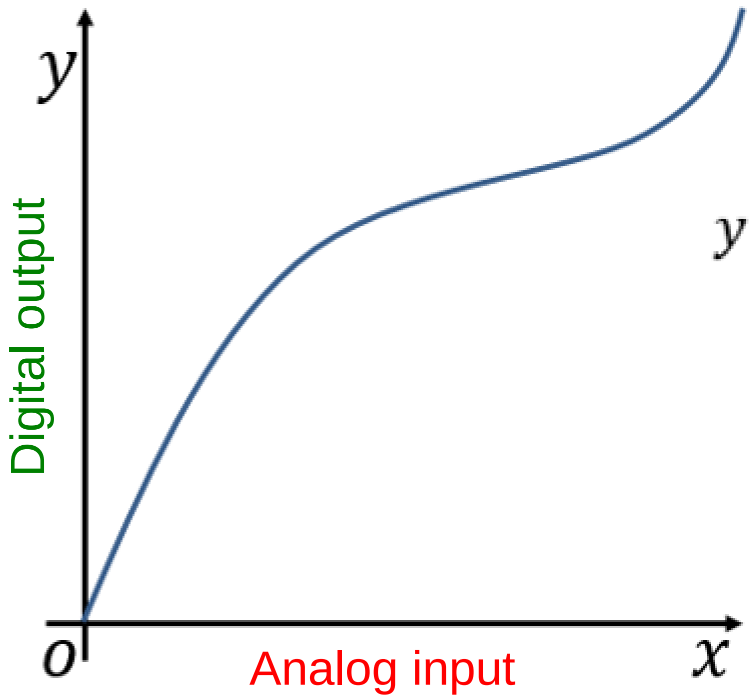
Cosine Input & FFT Analysis



I/O Characteristic Modeling of Modulator

Modeling by polynomial approximation

✓ **Assumption**: I/O characteristics are continuous in the AD modulator.



$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

3rd order model for simplicity :

$$y(t) = a_1 x(t) + a_3 x(t)^3$$

Polynomial Coefficient Estimation

Analog cosine input :

$\boldsymbol{x(t) = A\cos(\omega t)}$

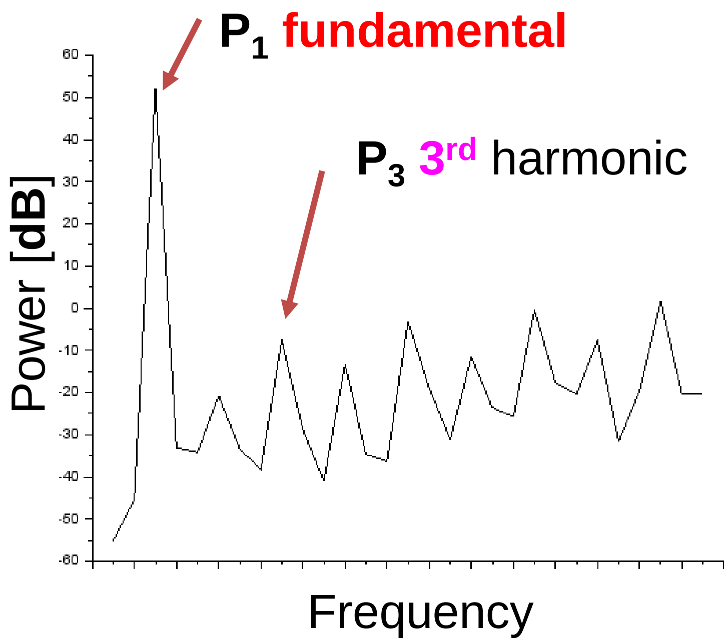
Modulator **1-bit** output stream

FFT

Measure **fundamental** &
3rd harmonic power

Estimate a_1, a_3 :

$$y(t) = a_1 x(t) + a_3 x(t)^3$$



Fundamental / HD3 and Polynomial Coefficients

Cosine input :

$$x(t) = A\cos\omega t$$

Output characteristic model :

$$y(t) = a_1 x(t) + a_3 x(t)^3$$

$$y(t) = a_1 A\cos\omega t + a_3 (A\cos\omega t)^3$$

$$(a_1 A + \frac{3}{4} a_3 A^3)\cos\omega t \quad + \quad \frac{1}{4} a_3 A^3 \cos 3\omega t$$

Fundamental

$$a_1 A + \frac{3}{4} a_3 A^3$$

HD3

$$\frac{1}{4} a_3 A^3$$

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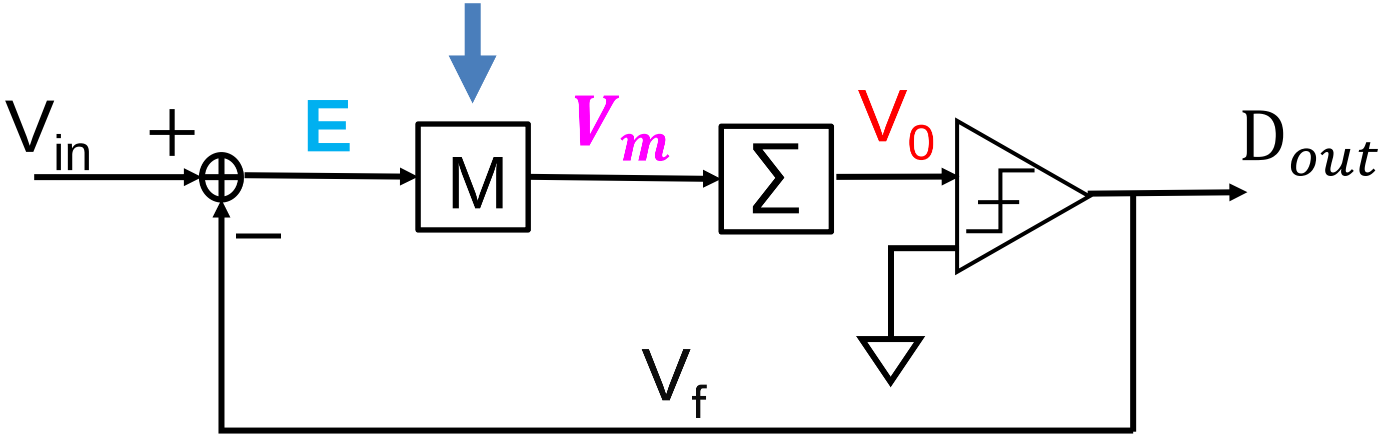
Simulation Model

3rd-order nonlinearity model

$V_m = E - k * E^3$ (k>0)

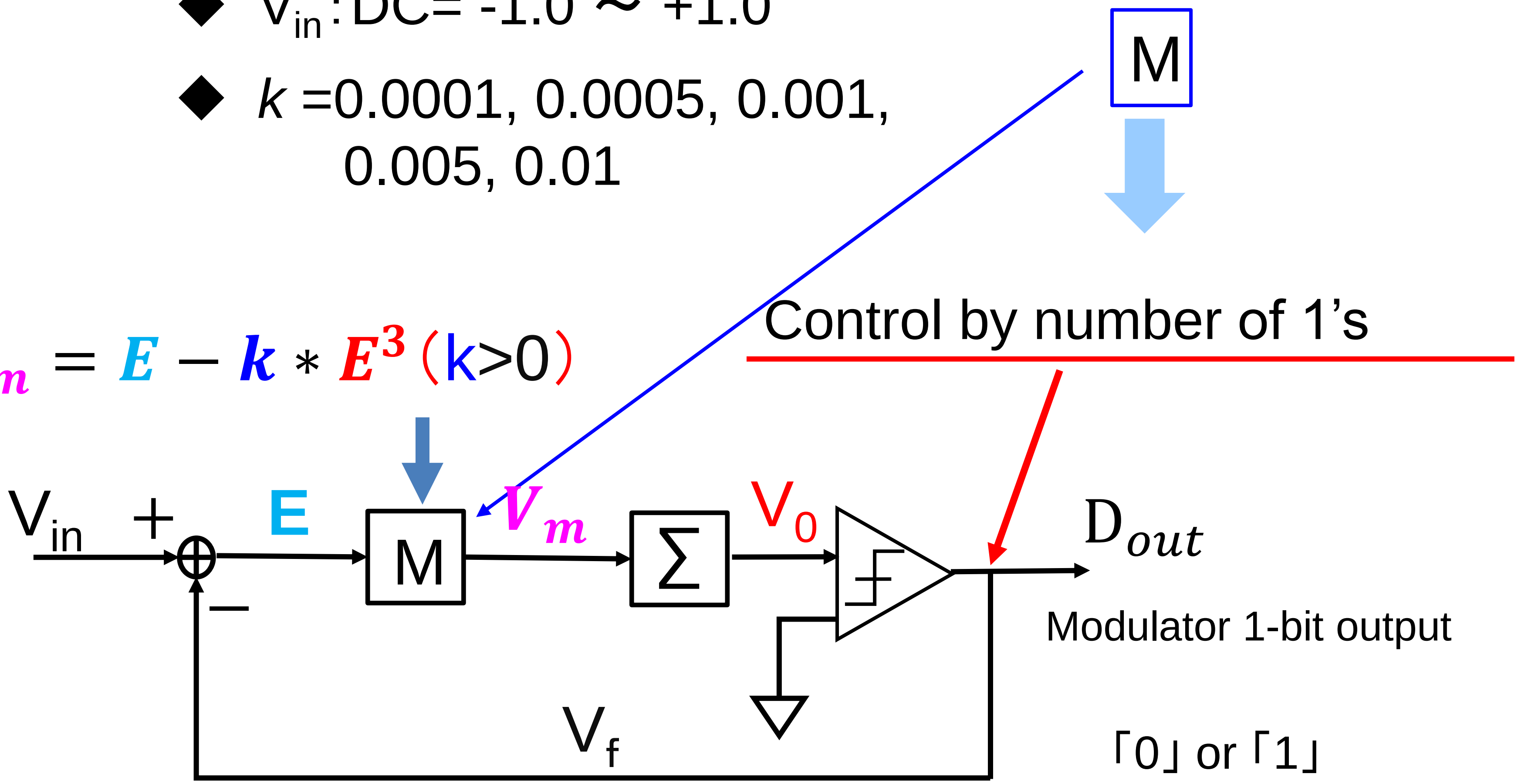


1st-order modulator

DC Input Simulation

- Number of data ：N=2^{20}
- V_{in}：DC= -1.0 ～ +1.0
- *k* =0.0001, 0.0005, 0.001, 0.005, 0.01

$$V_m = E - k * E^3 \quad (k>0)$$



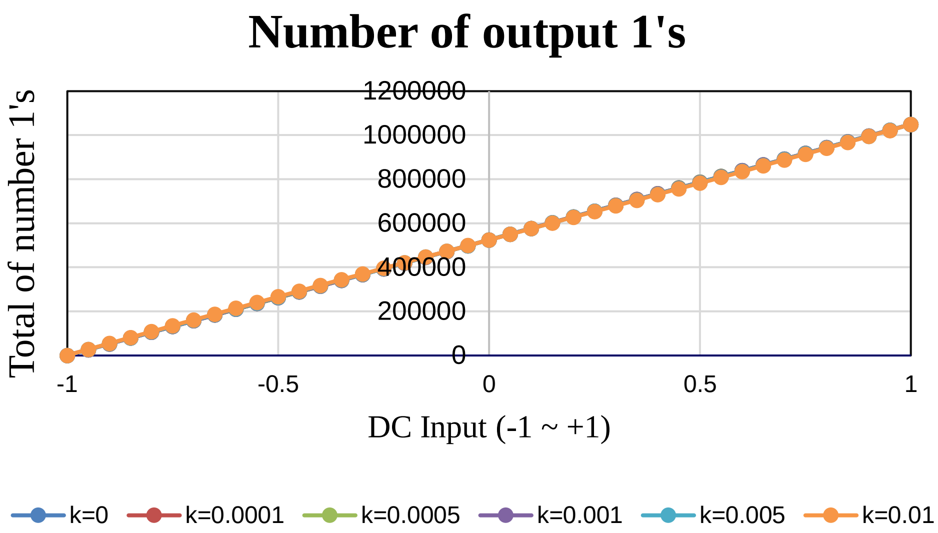
1st -order modulator

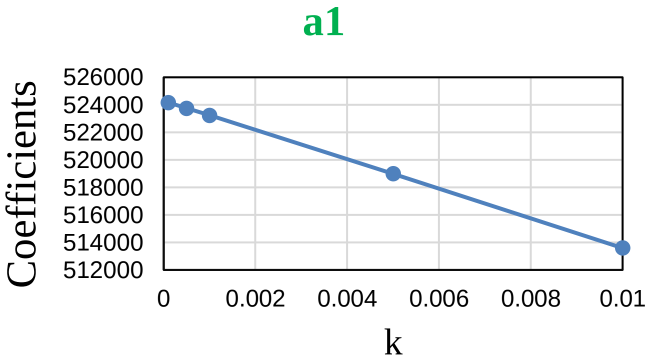
DC Input Simulation Result

Number of modulator output: N = 2^{20}

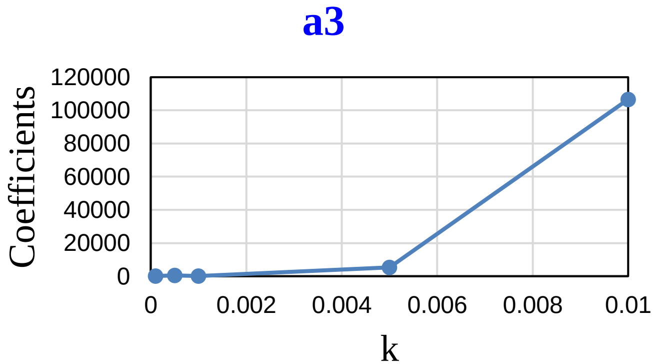
DC characteristic curve fitting

$$y = a_3 \times x^3 + a_1 \times x$$



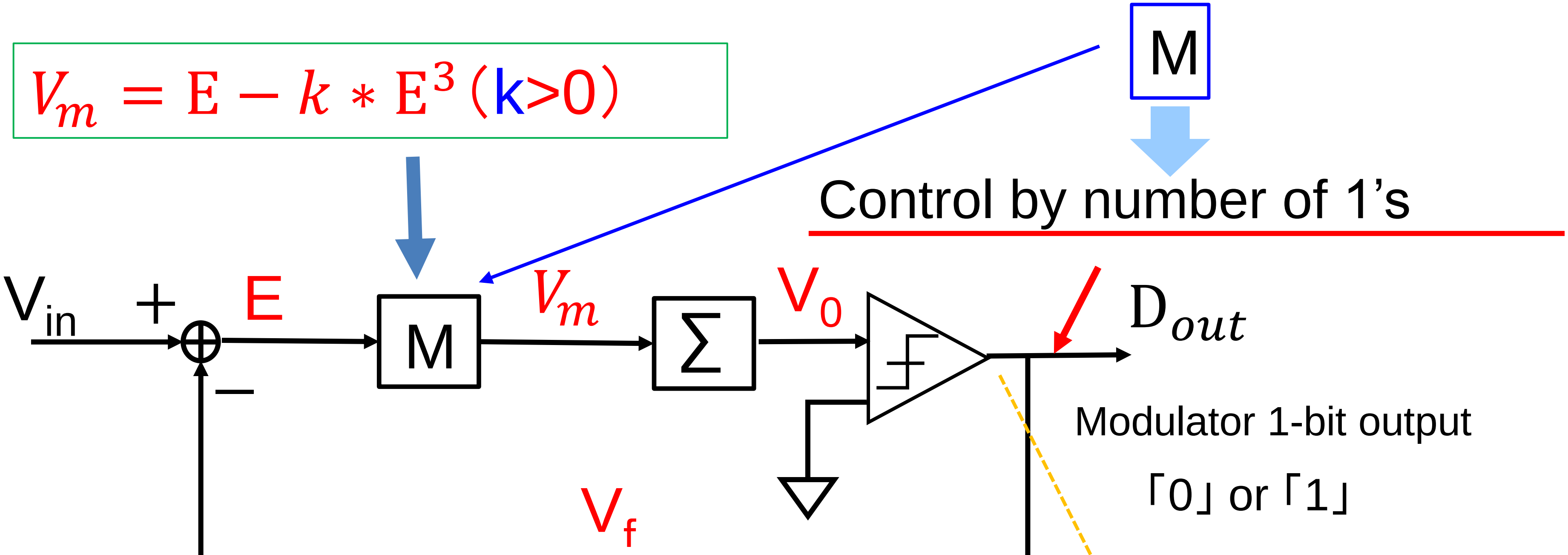


k	a_3	a_1
0.0001	104.84	524180
0.0005	524.48	523760
0.0010	1050.5	523240
0.0050	5282.5	519000
0.0100	10643.0	513610



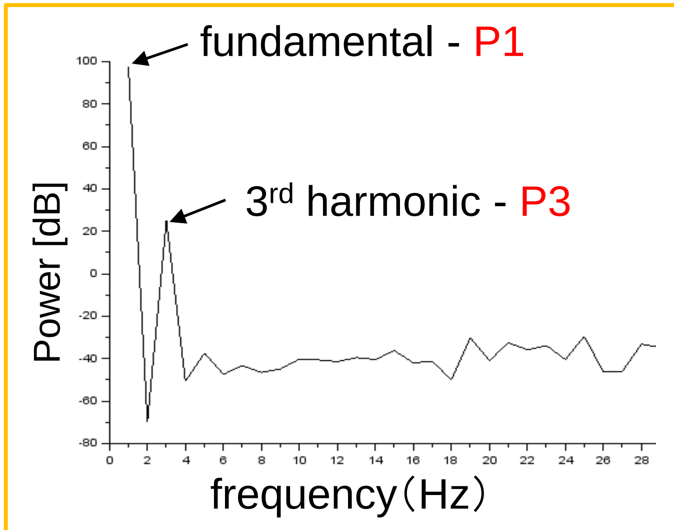
Cosine Input Simulation Configuration

$V_m = \mathrm{E} - k * \mathrm{E}^3$ (k>0)

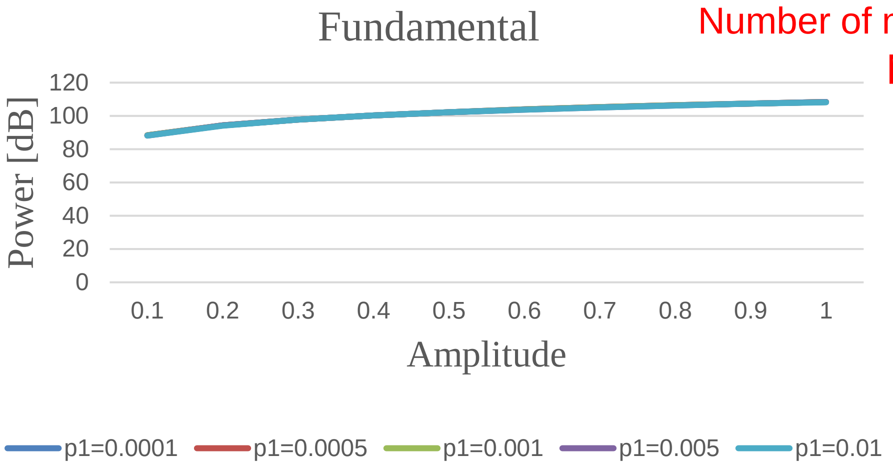


1st-order modulator

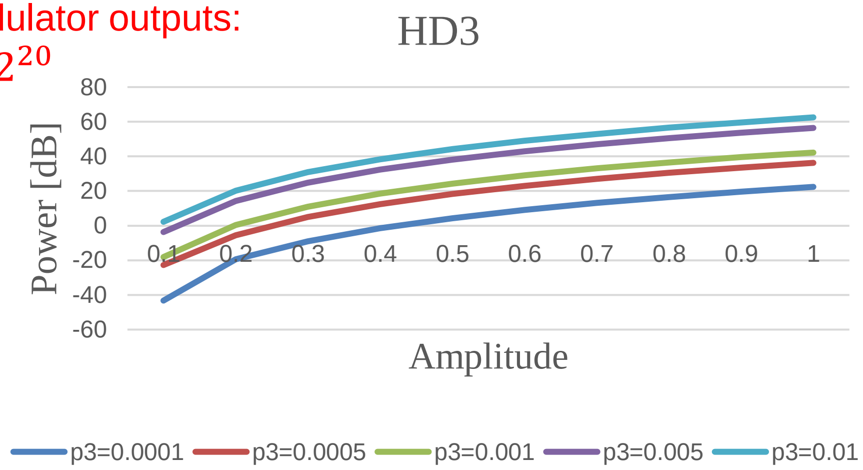
- Number of data : $N=2^{20}$
- V_{in} : $A\cos(\omega t)$ $(A = 0.1\sim1)$
- k=0.0001, 0.0005, 0.001, 0.005, 0.01



Cosine Input Simulation Result

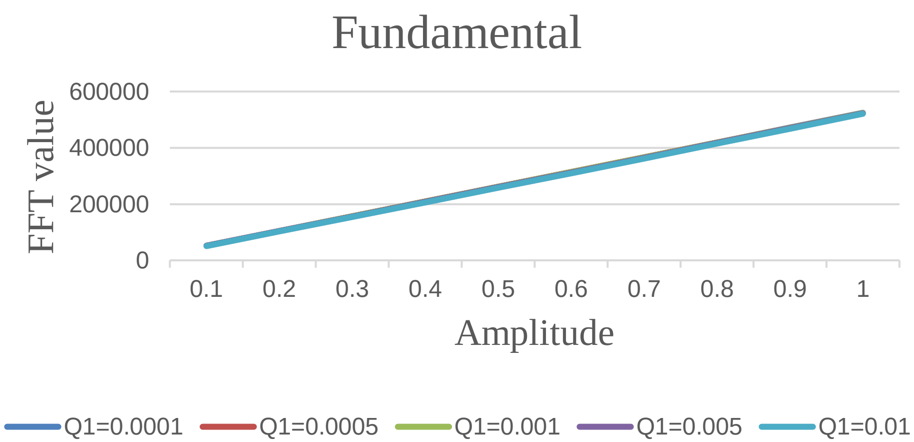


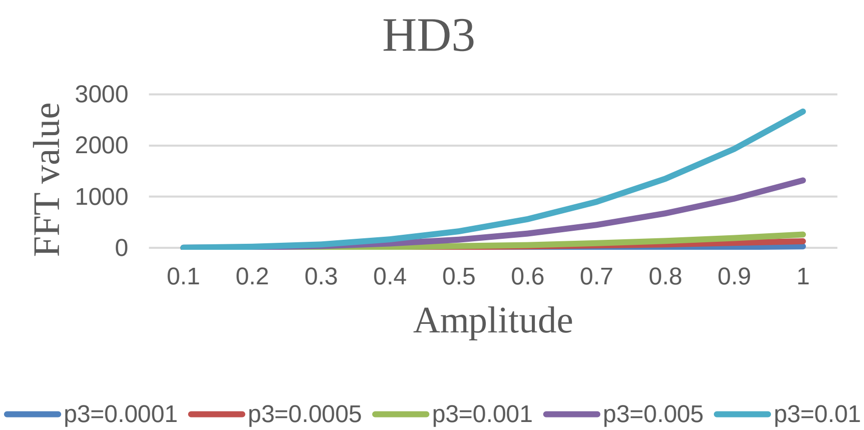
Number of modulator outputs:
$N=2^{20}$



Power = $20\log(FFT_{value}) - 6.02$ [dB]

FFT result





Find Spectrum Power from DC Characteristics

- **1st - order** modulator
- Number of 1-bit output data : N = $\mathbf{2^{20}}$

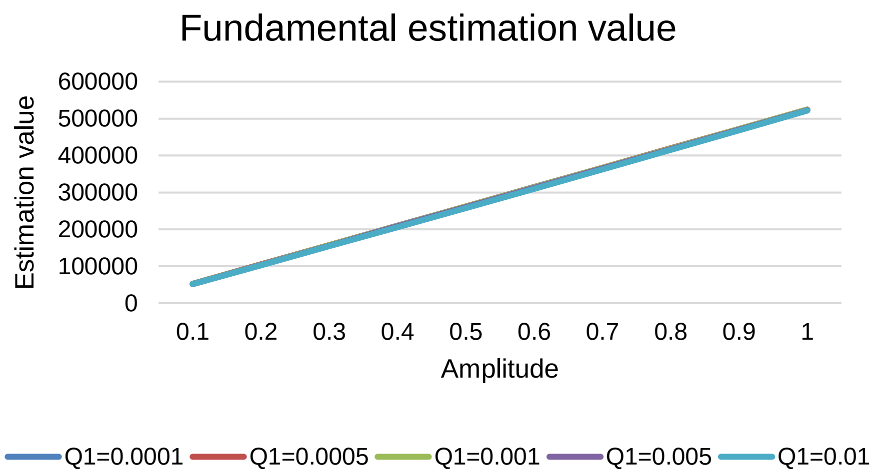
k	a_3	a_1
0.0001	104.84	524180
0.0005	524.48	523760
0.0010	1050.5	523240
0.0050	5282.5	519000
0.0100	10643.0	513610

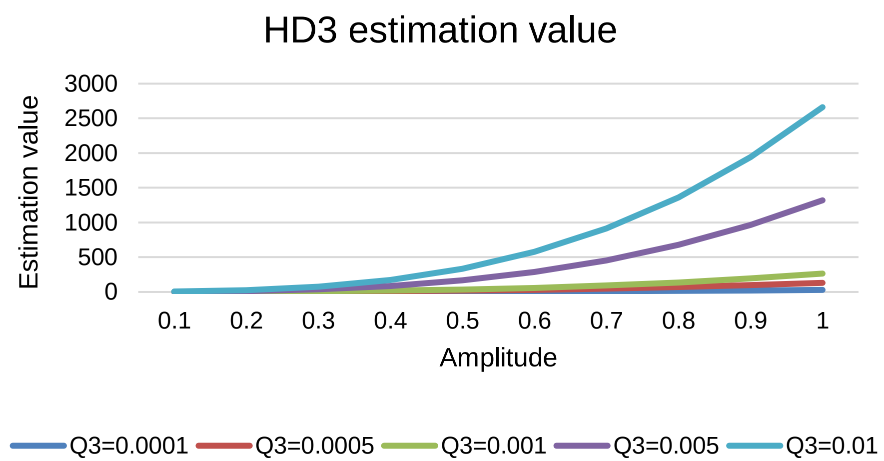
By nonlinearity

Fundamental : $a_1 A + \frac{3}{4} a_3 A^3$

HD3 : $\frac{1}{4} a_3 A^3$

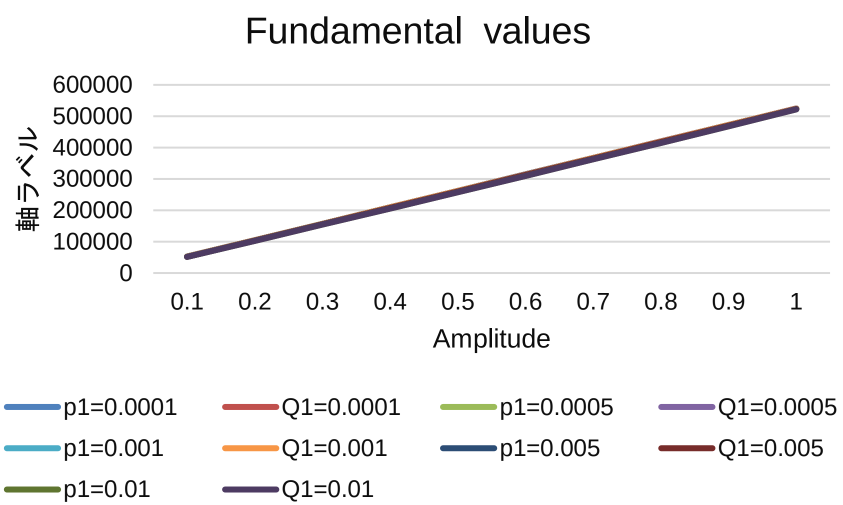
A : amplitude

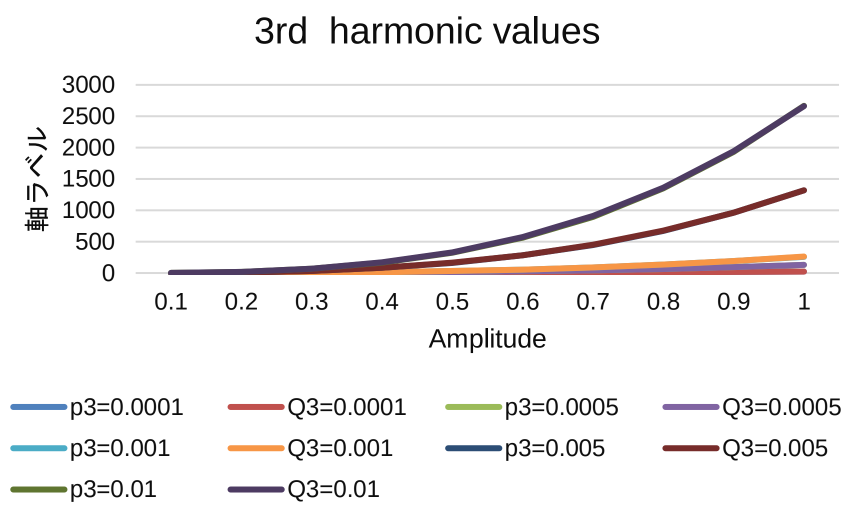




N=2^{20}

Comparison between Estimated and FFT Values

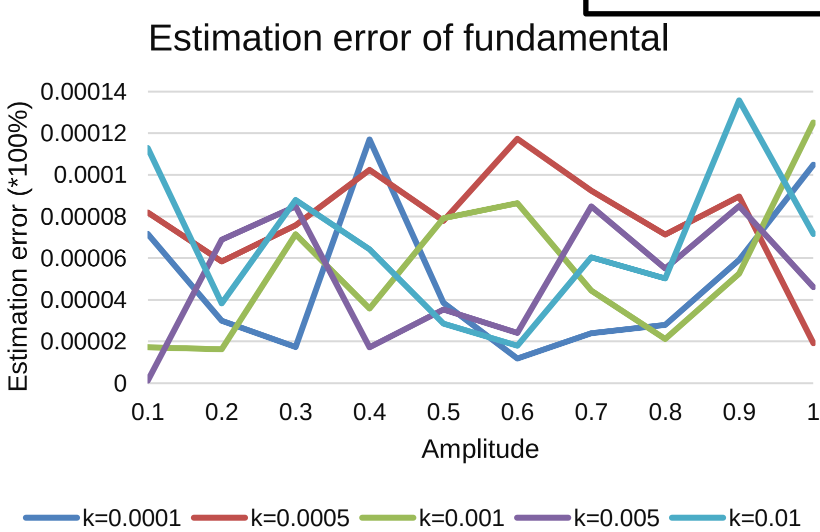


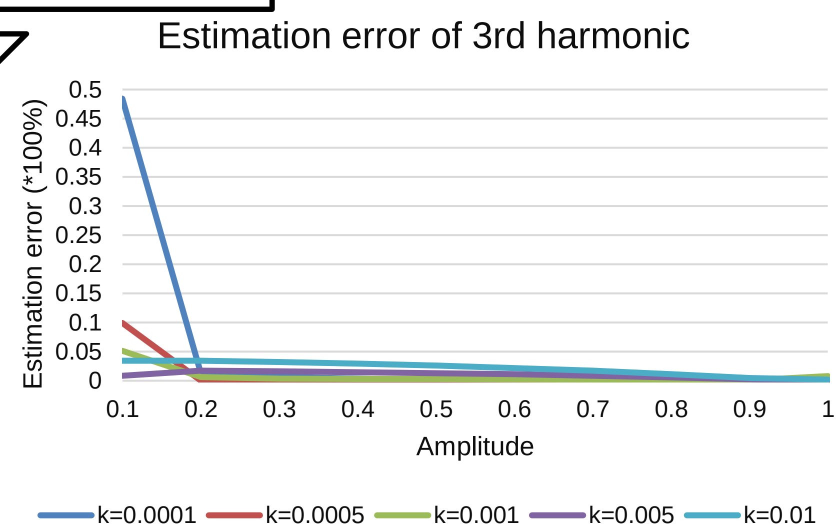


P_1 : fundamental obtained by FFT
Q_1 : estimated **fundamental**

$$\text{Error} = \frac{Q_{\text{values}} - P_{\text{values}}}{Q_{\text{values}}}$$

P_3 : HD3 obtained by FFT
Q_3 : estimated **HD3**



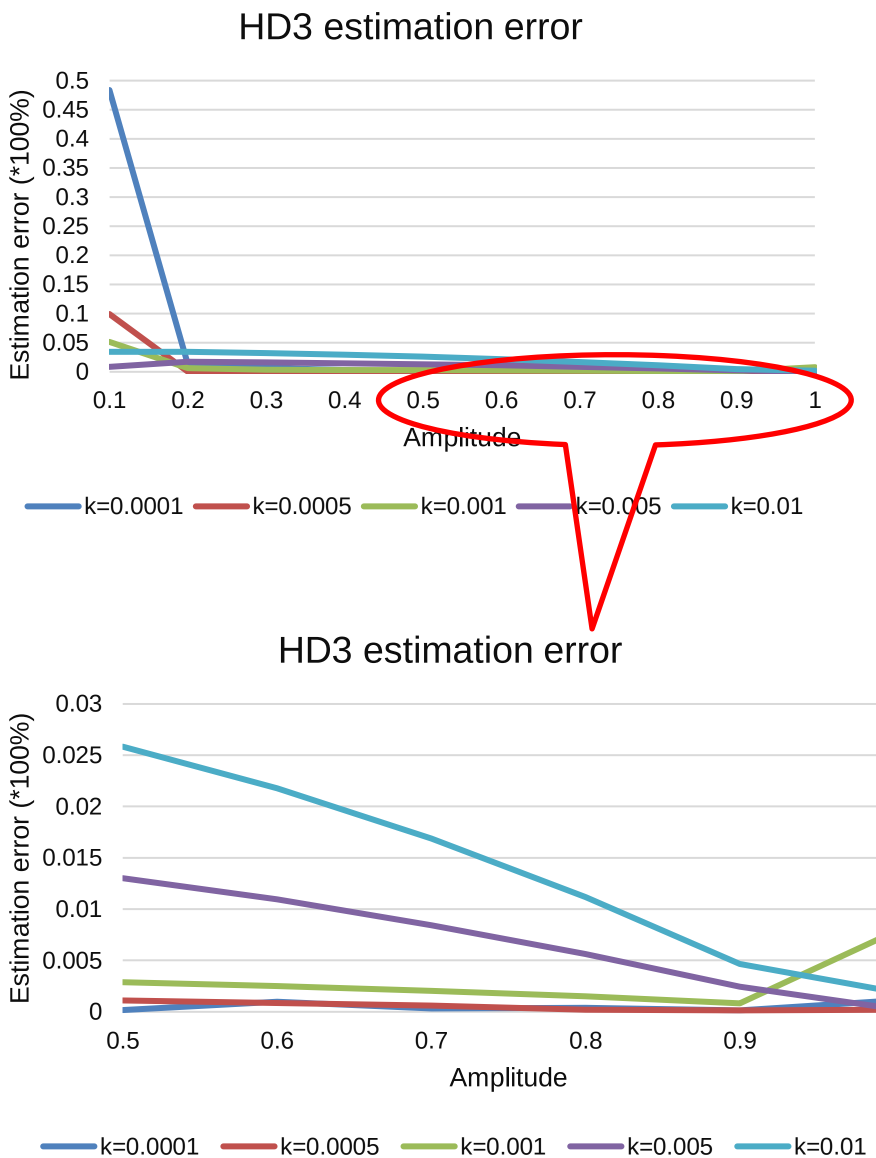


Accurate Estimation Condition for HD3

1st -order modulator

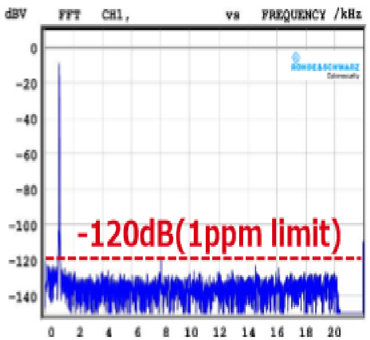
Good condition
HD3

Amplitude = 0.9
Error = 0.0123%
(k = 0.0005)



DUT Measurement Result

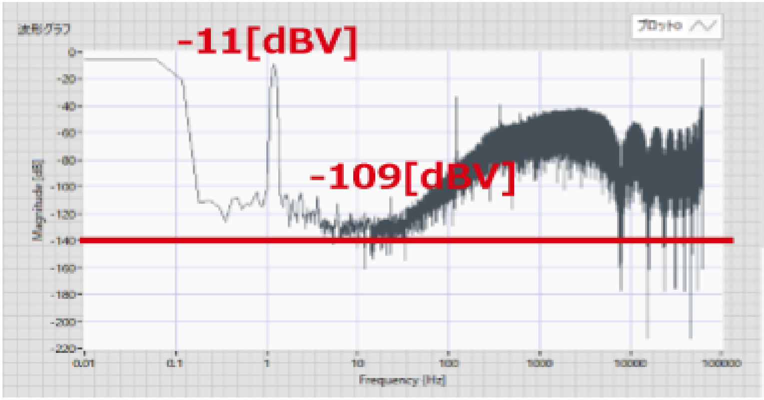
➢ **Measurements results from ROHM semiconductor company**



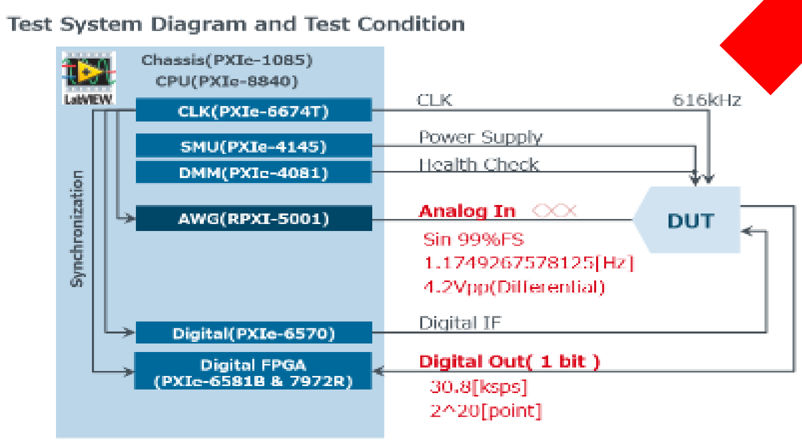
Output : 1kHz 44.1ksps
THD : 122dB(~Fifth Harmonics)
SN : 132dB(Filter:20kHzLPF)

Meet the requirements

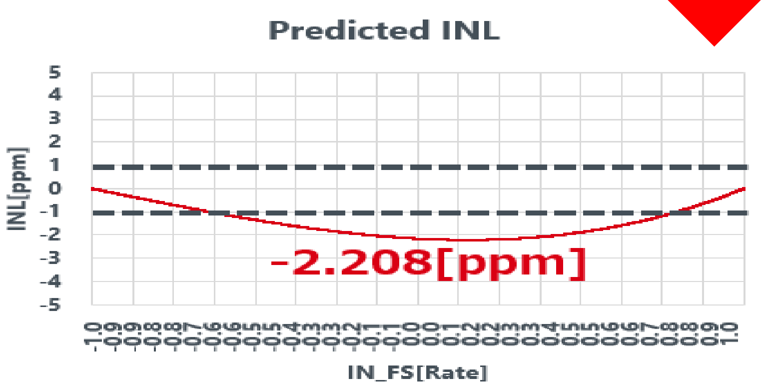
Signal from our developed **AWG**
(**AWG**: Arbitrary Waveform Generator)



Experimental result of the modulator output FFT



Use of NI PXI system for experiment



Obtained INL prediction with the proposed method

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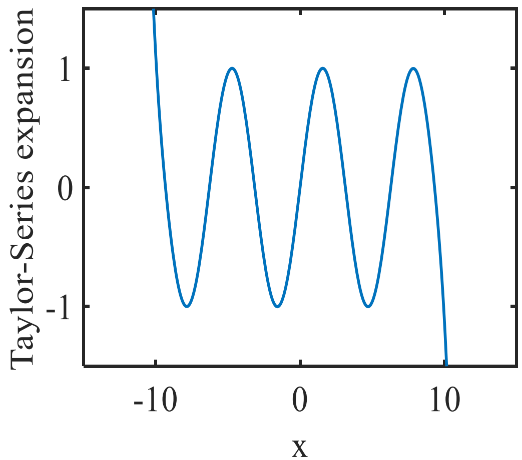
Taylor Series Expansion

- **Re-write a smooth function as infinite sum of polynomial terms.**

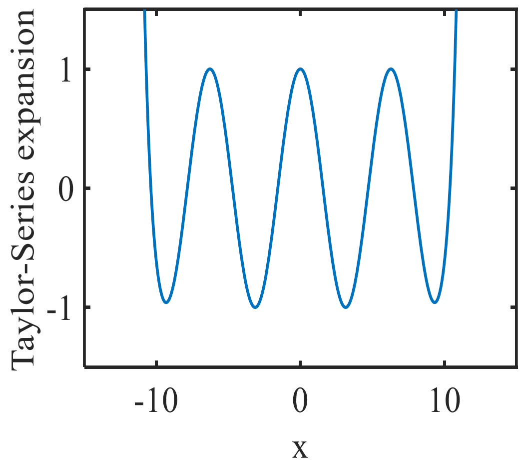
Function $\boldsymbol{f(x)}$ for $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{(f)''(a)}{2!}(x-a)^2 + \cdots + \frac{(f)^n(a)}{n!}(x-a)^n + \cdots$$

Convergence range: $\alpha < x < \beta$



$\mathbf{sin}(\boldsymbol{x})$



$\mathbf{cos}(\boldsymbol{x})$

Central value: $\boldsymbol{a = 0}$

Number of Taylor-series expansion: **20**.

Floating-Point Representation

Mantissa : **M** $(1 \leqq M < 2)$

Exponent : **E**

$$M \times 2^{E}$$

Decimal point

$$1.abcdef\cdots \times 2^{E}$$

Mantissa **Exponent**

$a, b, c, d, e, f, \cdots$: 0 or 1

IEEE-754 standard:

- ◆ Half-precision **16-bit**
- ◆ Single-precision **32-bit**
- ◆ Double-precision **64-bit**

31	30 – 23	22 – 0
S	E (Exponent)	M (Mantissa)

IEEE-754 **single-precision** floating-point format

Exponential Calculation

Floating-point binary : $X = M \times 2^{E}$

Exponential calculation of $\boldsymbol{X}$.

$$EXP = exp(X) = exp(M \times 2^{E})$$

$$(exp(M))^{2^{E}}$$

$\boldsymbol{exp(M)}$ calculation by Taylor-series expansion for specified accuracy.

Analysis of Taylor Expansion

Calculate exponential of mantissa : $\boldsymbol{exp(M)}$ $(1 \leq M < 2)$

$x = M$

$f(x) = exp(x)$ by Taylor expansion at $x = a$ $(1 \leq a < 2)$

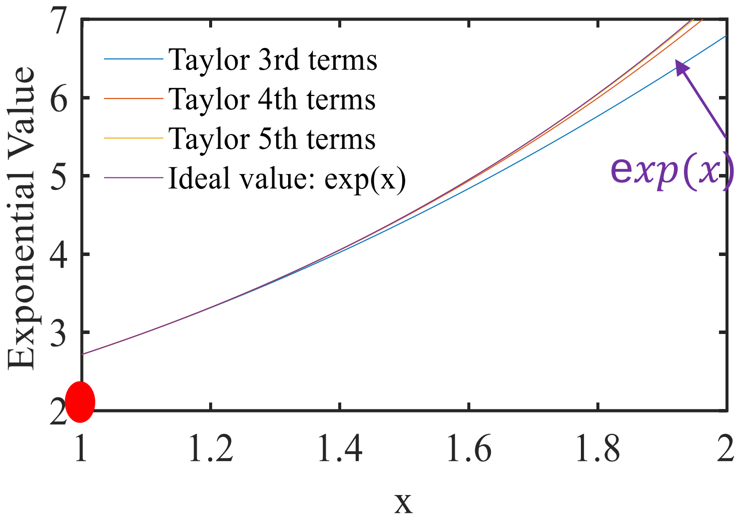
$$f(x) = exp(a) \times \left\{ 1 + q + \frac{1}{2}q^2 + \boxed{\frac{1}{6}}q^3 + \boxed{\frac{1}{24}}q^4 + \boxed{\frac{1}{120}}q^5 + \boxed{\frac{1}{720}}q^6 + \cdots \right\}$$

$\boldsymbol{q} = x - a$

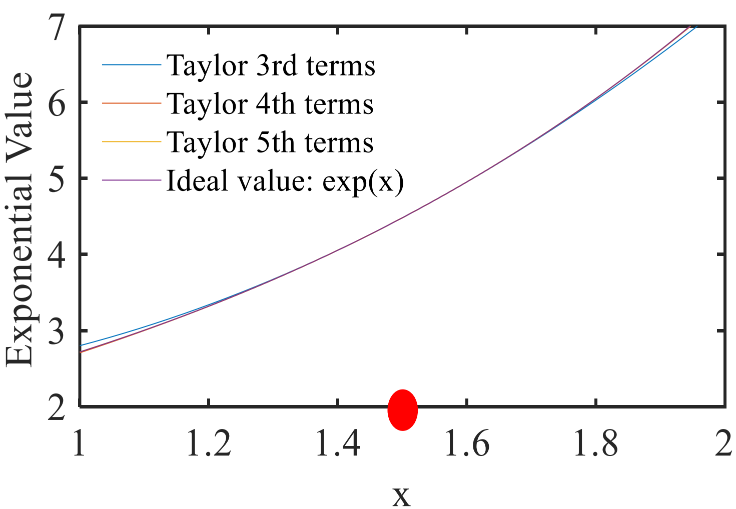
Coefficient values: stored in **LUT** in advance.

LUT : Look-Up Table

Mantissa Region Division



Taylor series expansion of $\boldsymbol{exp(x)}$ at center value *a* = **1**



at center value a = **1.5**

Divide and Conquer Method

1 region :

a = 1.5 $1 \le x < 2$

2 regions :

a = 1.25 $1 \le x < 1.5$

a = 1.75 $1.5 \le x < 2$

4 regions :

a = 1.125 $1 \le x < 1.25$

a = 1.375 $1.25 \le x < 1.5$

a = 1.625 $1.5 \le x < 1.75$

a = 1.875 $1.75 \le x < 2$

⋮

Outline

Definition of Accuracy

Example : $\frac{1}{2^{16}}$ accuracy

$$max \left| \frac{f(x) - t(n,x)}{f(x)} \right| \leq \frac{1}{2^{16}}$$

$f(x)$: Original function

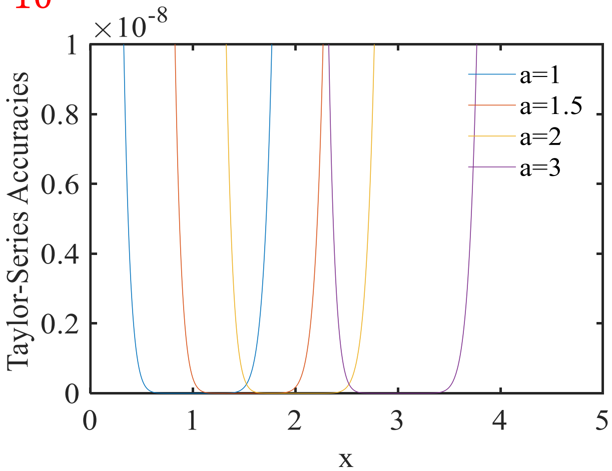
$t(n,x)$: Taylor expansion of n terms

Accuracy of $exp(x)$ Taylor Expansion

Number of Taylor expansion terms : 10

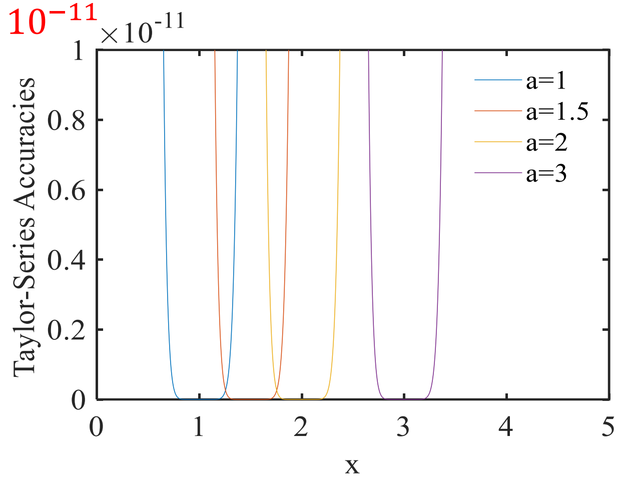
Accuracy : 10^{-8}

Errors



Accuracy : 10^{-11}

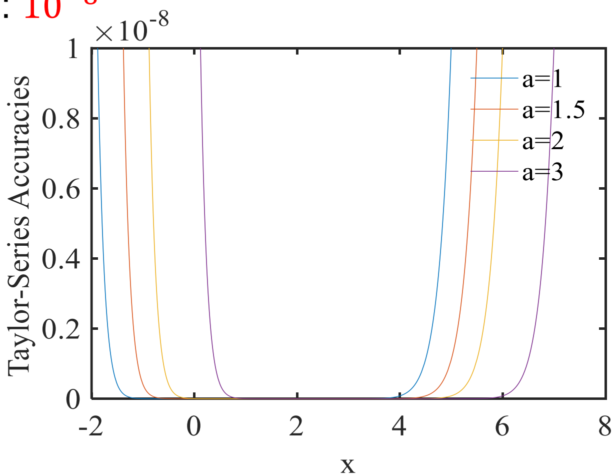
Errors



Number of Taylor expansion terms : 20

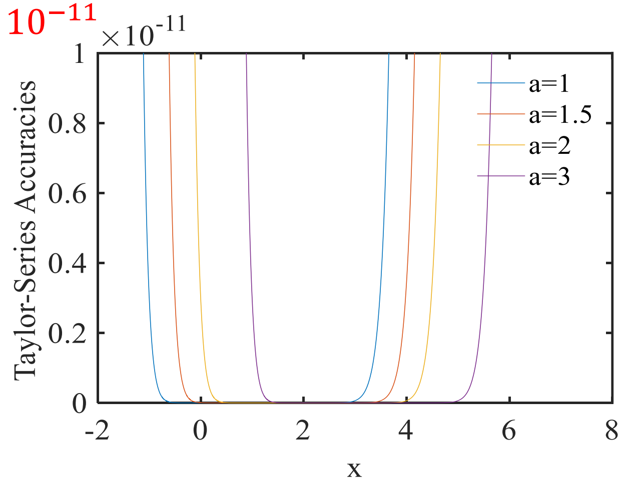
Accuracy : 10^{-8}

Errors



Accuracy : 10^{-11}

Errors



One-Region Case

Use Taylor series expansion equation :

$f(x) = exp(x) \qquad (1 \leq x < 2)$

Mantissa represented by binary decimal point.

Specified accuracy

Taylor-series expansion \ Accuracy		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) $M =$ 1.xxxxxx⋯ $1 \leq M < 2$	$a = 1.5$	4	7	8	9	11

Taylor series expansion at center value $a = 1.5$

Number of Taylor expansion terms to meet specified accuracy.

Two-Region Case

Use Taylor series expansion equation :

$f(x) = exp(x) \quad (1 \le x < 2)$

(0 or 1) of the **first decimal** place of Mantissa.

Taylor-series expansion \ Accuracy		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) M_D = 1.0xxxxx··· $1 \le M_D < 1.5$	$a = 1.25$	3	5	6	7	9
(ii) M_D = 1.1xxxxx··· $1.5 \le M_D < 2$	$a = 1.75$	3	5	6	7	9

Four-Region Case

Use Taylor series expansion equation :

$f(x) = exp(x) \quad (1 \leq x < 2)$

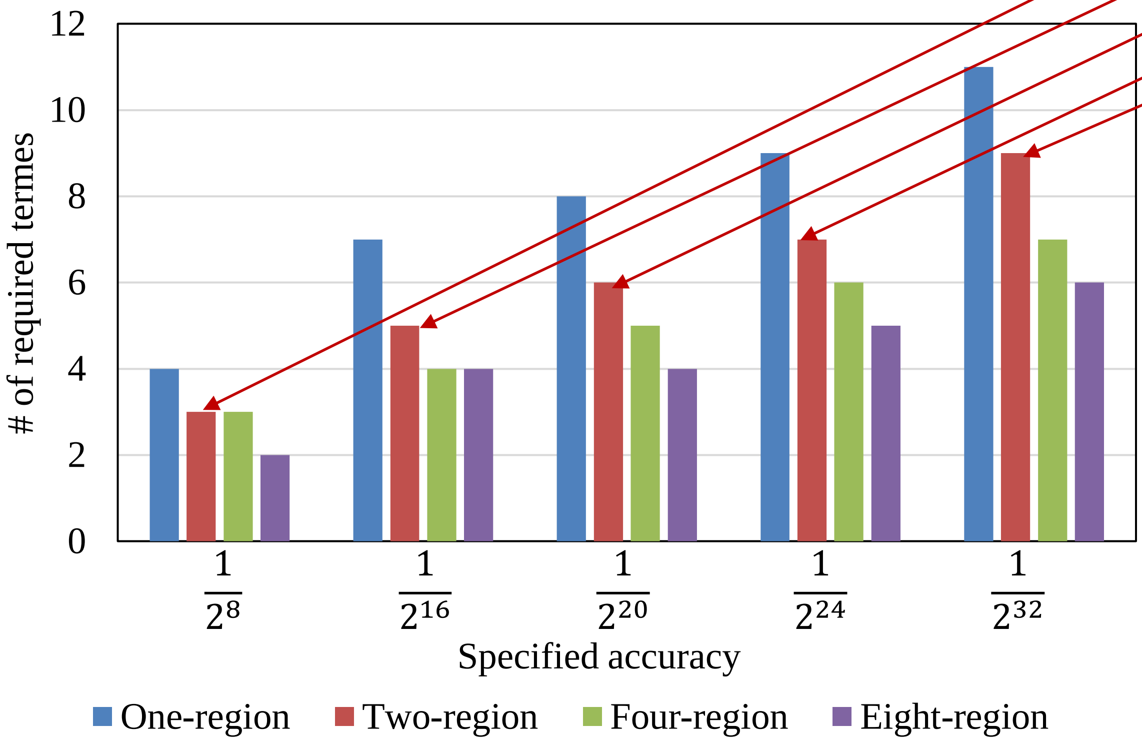
(00, 01, 10 or 11)
of the **first two decimal** places of Mantissa.

Taylor-series expansion \ Accuracy		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) M = 1.00xxxx··· $1 \leq M_D < 1.25$	a=1.125	3	4	5	6	7
(ii) M = 1.01xxxx··· $1.25 \leq M_D < 1.5$	a=1.375	3	4	5	6	7
(iii) M = 1.10xxxx··· $1.5 \leq M_D < 1.75$	a=1.625	3	4	5	6	7
(iv) M = 1.11xxxx··· $1.75 \leq M_D < 2$	a=1.875	3	4	5	6	7

Comparison of Number of Required Terms

Comparison of
required number of **terms**
for **different number** of **region divisions**

Taylor-series expansion \ Accuracy		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) M_D = 1.0xxxxx··· $1 \le M_D < 1.5$	$a = 1.25$	3	5	6	7	9
(ii) M_D = 1.1xxxxx··· $1.5 \le M_D < 2$	$a = 1.75$	3	5	6	7	9



Lager number of divided regions

Number of terms **reduced;**
LUT size becomes **larger.**

Outline

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4. **Conclusion**

Calculation Complexity

- In case of Taylor expansion **5** terms :

$$f_5(x) = exp(a) \times \left\{1 + (x-a) + \frac{(x-a)^2}{2} + \frac{(x-a)^3}{6} + \frac{(x-a)^4}{24}\right\}$$

- $exp(a)$ values : **Stored** in **LUT** and **read**.

y = x−a Subtraction: **1 time** z = y^2 Multiplication: **1 time**

$$f_5(x) = \exp(\mathrm{a}) \times \left(1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24}\right)$$

$$= \exp(a) \times \left\{1 + y + \frac{z}{2} \times (1 + \frac{y}{3} + \frac{z}{12})\right\}$$

Multiplication: **5** times
Addition / Subtraction : **4** times

Total : Multiplication: **6** times
Addition / Subtraction : **5** times

Number of Operations

Number of terms versus number of operations in Taylor expansion

Taylor expansion of $f(x) = exp(x)$ can be calculated with small number of **Mul/ Add/Sub** operations.

Terms of Taylor expansion	Multiplication	Addition / Subtraction
3	3	3
4	5	4
5	6	5
6	8	6
7	9	7
8	10	8

In case of 5 terms

LUT Size

$$f_5(x) = \boxed{exp(a)} \times \left\{1 + (x-a) + \frac{(x-a)^2}{2} + \frac{(x-a)^3}{6} + \frac{(x-a)^4}{24}\right\}$$

Stored in **LUT**

4 - region case $\longrightarrow$ LUT size of **4** words

Address (M=1.*ab*⋯)	LUT data
00	$Exp(a)$ for a = 1.125
01	$Exp(a)$ for a = 1.357
10	$Exp(a)$ for a = 1.625
11	$Exp(a)$ for a = 1.875

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Conclusion

- **High resolution, low speed ΔΣ ADC** linearity
  short time testing algorithm

  - Polynomial modeling of
    modulator **input / output characteristics**
  - FFT of modulator **1-bit output** stream for cosine input
  - Estimate polynomial coefficients
    from **fundamental** and **harmonic** powers

- **Verified** by **simulation** and **experiments**

Drastic **testing time reduction**:
104 **days** → **32 seconds**

Conclusion

- **Mantissa calculation** of **exponential function** with Taylor-expansion

Divide and Conquer Method

- Number of **divided mantissa regions** becomes **larger**

⬇

➢ Number of Taylor expansion terms ➡ **smaller**

➢ **LUT size** ➡ **larger**

⬇

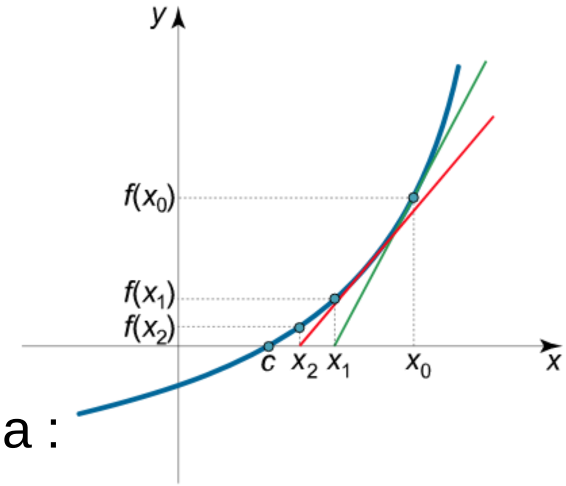
Optimal hardware configuration

Thank you for listening !

Appendix

Newton's method

Newton's method step:



Fist, Start with an initial approximation x_0 close to c.

Second, Determine the next approximation by the formula :

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

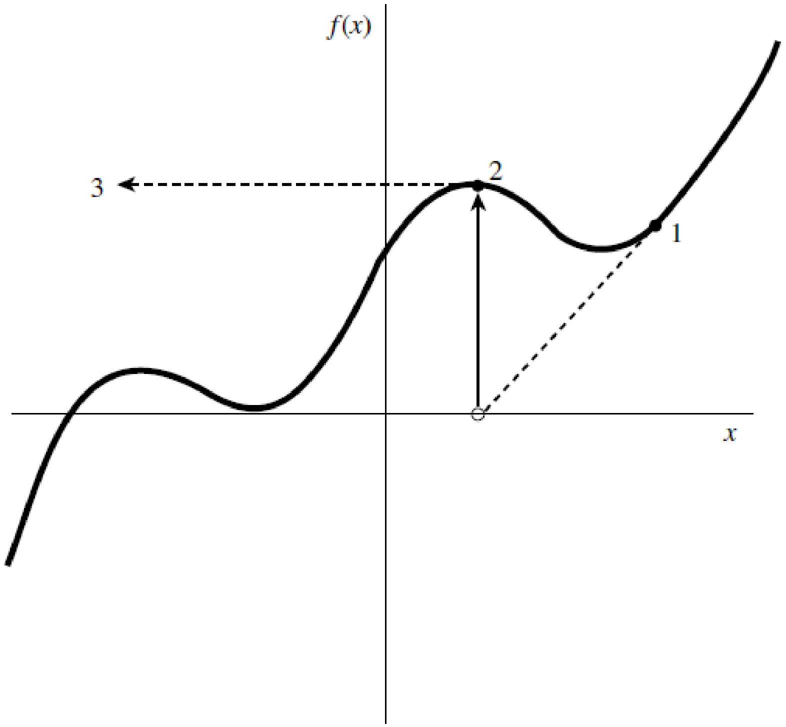
Third, Continue the iterative process using the formula :

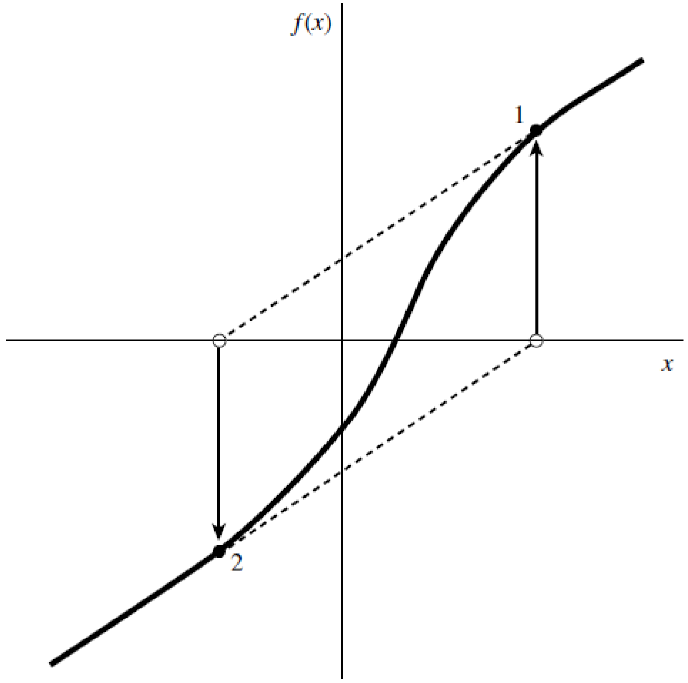
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Last, until the root is found to the desired accuracy.

- Poor global convergence properties
- Dependent on initial guess
  - May be too far from local root
  - May encounter a zero derivative
  - May loop indefinitely

Examples of disadvantages





On the left, we have Newton's Method finding a local maxima, in such cases the method will shoot off into negative infinity.

Newton's Method has entered an infinite cycle. Better initial guesses may be able to alleviate this problem.

Another Decimal Point Position

Change the decimal point position of the mantissa

Mantissa: M　　　　Exponent: E

$M \times 2^{E}$

Original decimal point

$1.abcdef \cdots \times 2^{E}$

Mantissa　　Exponent

$1 \leqq M < 2$

$M \times 2^{E}$

New decimal point

$0.1abcdef \cdots \times 2^{E}$

Mantissa　　Exponent

$0.5 \leqq M < 1$

Ex :1011001 (binary) = 89 (decimal)

Binary representation : 0.1011001×2^{111}

Decimal representation : $0.6953125 \times 2^{7} = 89$

Eight-Region Case

Check the values (000, 001,…, 111)
of the first three decimal places of Mantissa.

Taylor-series expansion \ Accuracy		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) M = 1.000xxxx⋯ $1 \leq M_D < 1.125$	a=1.0625	2	4	4	5	6
(ii) M = 1.001xxxx⋯ $1.125 \leq M_D < 1.25$	a = 1.1875	2	4	4	5	6
(iii) M = 1.010xxxx⋯ $1.25 \leq M_D < 1.375$	a=1.3125	2	4	4	5	6
(iv) M = 1.011xxxx⋯ $1.375 \leq M_D < 1.5$	a=1.4375	2	4	4	5	6
(v) M = 1.100xxxx⋯ $1.5 \leq M_D < 1.625$	a=1.5625	2	4	4	5	6
(vi) M = 1.101xxxx⋯ $1.625 \leq M_D < 1.75$	a=1.6875	2	4	4	5	6
(vii) M = 1.110xxxx⋯ $1.75 \leq M_D < 1.875$	a=1.8125	2	4	4	5	6
(viii) M = 1.111xxxx⋯ $1.875 \leq M_D < 2$	a=1.9375	2	4	4	5	6

Exponential Calculation in Different Ranges

$-2 \le x < -1$ case:

Use Taylor series expansion equation : $f(x) = \exp(x)$ $(-2 \le x < -1)$

$-2 \le x < -1$ in One-region case						
Taylor-series expansion \ Accuracy		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) $M =$ 1.xxxxxx··· $-2 \le M < -1$	$a = -1.5$	4	7	8	9	10

$0.5 \le x < 1$ case:

Use Taylor series expansion equation : $f(x) = \exp(x)$ $(0.5 \le x < 1)$

$0.5 \le x < 1$ in One-region case						
Taylor-series expansion \ Accuracy		$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	$\frac{1}{2^{20}}$	$\frac{1}{2^{24}}$	$\frac{1}{2^{32}}$
(i) $M =$ 1.xxxxxx··· $0.5 \le M < 1$	$a = 0.75$	3	5	6	7	9