Distributed Arithmetic for Taylor-Series Expansion

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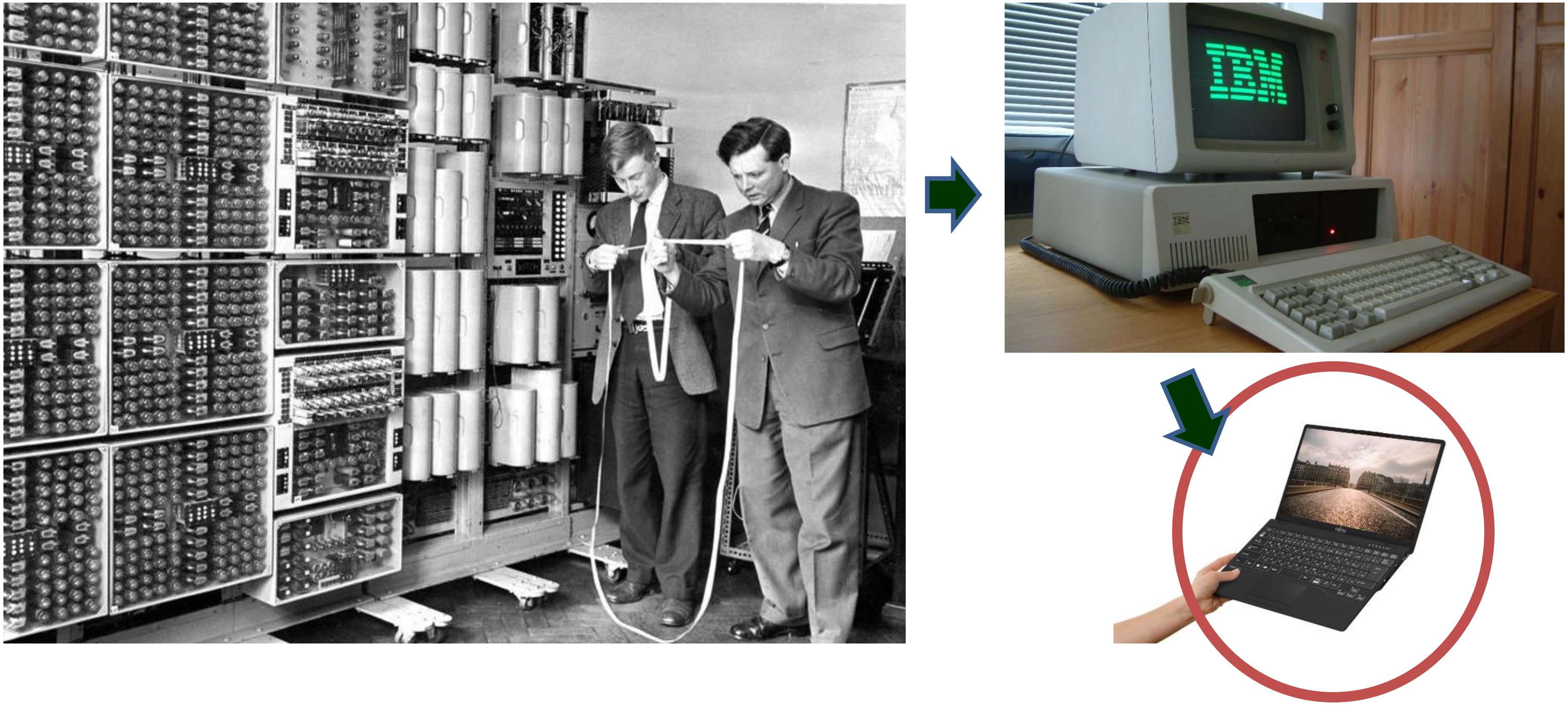
Changes in Electronic Devices

PC, Phone…Electronic Devices

Smaller



Research Objective

Find algorithm that can

Make small circuit to

Calculate Taylor-Series Expansion

$$\boldsymbol{f(x) = f(\alpha) + \frac{f'(\alpha)}{1!}(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2 + \frac{f'''(\alpha)}{3!}(x-\alpha)^3 + \cdots}$$

Contents

- **Outline**
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Taylor-Series Expansion and DA (Problem 1)

DA: Distributed Arithmetic

Taylor-Series Expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9$$

$$\Downarrow$$

$$f(x) = a_0 + \underbrace{x^1}_{}\left(a_1 + \underbrace{x^1}_{}\left(a_2 + \underbrace{x^1}_{}\left(a_3 + \underbrace{x^1}_{}\left(a_4 + \underbrace{x^1}_{}\left(a_5 + \underbrace{x^1}_{}\left(a_6 + \underbrace{x^1}_{}\left(a_7 + \underbrace{x^1}_{}\left(a_8 + \underbrace{a_9 x^1}_{}\right)\right)\right)\right)\right)\right)\right)\right)$$

9 Multiplications

$\Downarrow$ Distributed Arithmetic

Use Multiplier → $\underline{x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9}$ for Look Up Table

8 Multiplications

Large number of multiplications

Taylor-Series Expansion and DA (Problem 2)

DA: Distributed Arithmetic

Taylor-Series Expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9$$

⇓ Distributed Arithmetic

$x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9$ for Look Up Table

Look Up Table

x^0	x^1	x^2	…	x^9	Output Data
0	0	0	…	0	0
1	0	0	…	0	a_0
0	1	0	…	0	a_1
…	…	…	…	…	…
1	1	1	…	1	$a_0 + a_1 + a_2 + \cdots + a_9$

2^{10} ↓ **1024**

10

Large size LUT

Our Approach

$$f(x) = \underbrace{(a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8)}_{\text{DA}} + x^1\underbrace{(a_1 + a_3x^2 + a_5x^4 + a_7x^6 + a_9x^8)}_{\text{DA}}$$

Multiplication: x^2, x^4, x^6, x^8, $x^1(a_1 + \cdots + a_9x^8)$

5 times

Look Up Table (Left DA)

x^0	x^2	x^4	x^6	x^8	Output Data
0	0	0	0	0	0
1	0	0	0	0	a_0
0	1	0	0	0	a_2
…	…	…	…	…	…
1	1	1	1	1	$a_0 + a_2 + a_4 + a_6 + a_8$

2^5 ↓ 32

5

Look Up Table (Right DA)

x^0	x^2	x^4	x^6	x^8	Output Data
0	0	0	0	0	0
1	0	0	0	0	a_1
0	1	0	0	0	a_3
…	…	…	…	…	…
1	1	1	1	1	$a_1 + a_3 + a_5 + a_7 + a_9$

Smaller number of multiplications

&

Smaller size memory

How to Reduce Multiplication?

Distributed Arithmetic

Multiplications / Additions

↓

LUT + Adder

Without Multiplier

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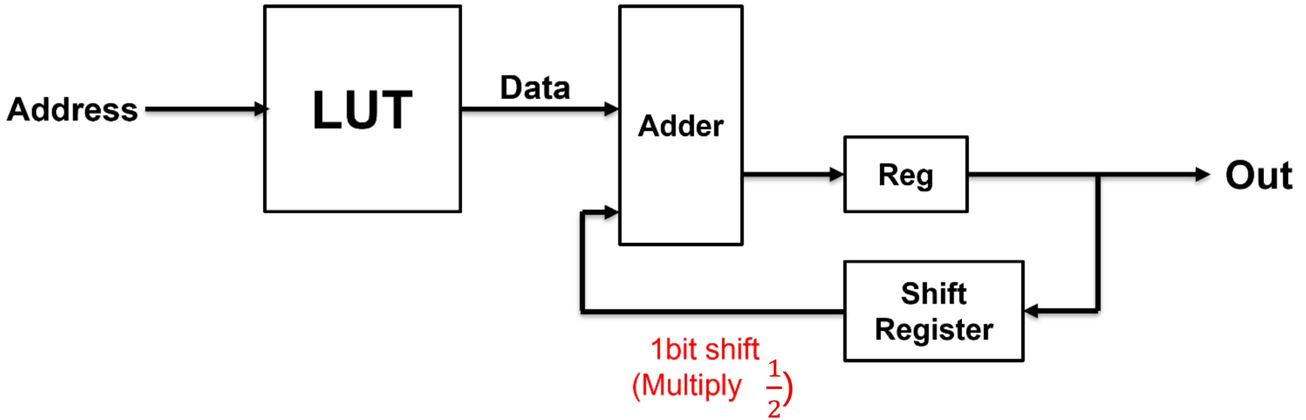
Flow of “Distributed Arithmetic”

- Look Up Table (LUT)
- Calculate x^n
- Bit-shift and Addition
- Correct the position of Decimal Point



Look Up Table (LUT)

$$f(x) = a_0 + a_1x + a_2x^2$$

LUT			
x^0	x^1	x^2	**Sum**
0	0	0	0
1	0	0	a_0
0	1	0	a_1
0	0	1	a_2
1	1	0	$a_0 + a_1$
1	0	1	$a_0 + a_2$
0	1	1	$a_1 + a_2$
1	1	1	$a_0 + a_1 + a_2$

Calculate x^n

$$f(x) = a_0 + a_1 x + a_2 x^2$$

If Decimal $x = 1.5$ → Binary $x = 1.1$

x^0	0	1.	0	0
x^1	0	1.	1	0
x^2	1	0.	0	1

x^0	x^1	x^2
0	0	1
1	1	0
0	1	0
0	0	1

Bit-shift and Addition

$$f(x) = a_0 + a_1 x + a_2 x^2$$

	x^0	x^1	x^2
MSB	0	0	1
	1	1	0
	0	1	0
LSB	0	0	1

LUT			
x^0	x^1	x^2	**Sum**
0	0	0	0
1	0	0	a_0
0	1	0	a_1
0	0	1	a_2
1	1	0	$a_0 + a_1$
1	0	1	$a_0 + a_2$
0	1	1	$a_1 + a_2$
1	1	1	$a_0 + a_1 + a_2$

Bit-shift and Addition (Start from LSB)

$$f(x) = a_0 + a_1 x + a_2 x^2$$

	x^0	x^1	x^2
MSB	0	0	1
	1	1	0
	0	1	0
LSB	0	0	1

LUT			
x^0	x^1	x^2	**Sum**
0	0	0	0
1	0	0	a_0
0	1	0	a_1
0	0	1	a_2
1	1	0	$a_0 + a_1$
1	0	1	$a_0 + a_2$
0	1	1	$a_1 + a_2$
1	1	1	$a_0 + a_1 + a_2$

a_2

Bit-shift and Addition (LSB + 1)

$$\boldsymbol{f(x) = a_0 + a_1 x + a_2 x^2}$$

	x^0	x^1	x^2
MSB	0	0	1
	1	1	0
	0	1	0
LSB	0	0	1

LUT			
x^0	x^1	x^2	**Sum**
0	0	0	0
1	0	0	a_0
0	1	0	a_1
0	0	1	a_2
1	1	0	$a_0 + a_1$
1	0	1	$a_0 + a_2$
0	1	1	$a_1 + a_2$
1	1	1	$a_0 + a_1 + a_2$

$$\left(\begin{matrix}\frac{1}{2} \times (a_2) \\ +(a_1)\end{matrix}\right)$$

Bit-shift and Addition (LSB + 2)

$$f(x) = a_0 + a_1 x + a_2 x^2$$

	x^0	x^1	x^2
MSB	0	0	1
	1	1	0
	0	1	0
LSB	0	0	1

LUT			
x^0	x^1	x^2	**Sum**
0	0	0	0
1	0	0	a_0
0	1	0	a_1
0	0	1	a_2
1	1	0	$a_0 + a_1$
1	0	1	$a_0 + a_2$
0	1	1	$a_1 + a_2$
1	1	1	$a_0 + a_1 + a_2$

$$\left(\begin{array}{c}\frac{1}{2}\times\left(\begin{array}{c}\frac{1}{2}\times(a_2)\\ +(a_1)\end{array}\right)\\ +(a_0 + a_1)\end{array}\right)$$

Bit-shift and Addition (End at MSB)

$$f(x) = a_0 + a_1 x + a_2 x^2$$

	x^0	x^1	x^2
MSB	0	0	1
	1	1	0
	0	1	0
LSB	0	0	1

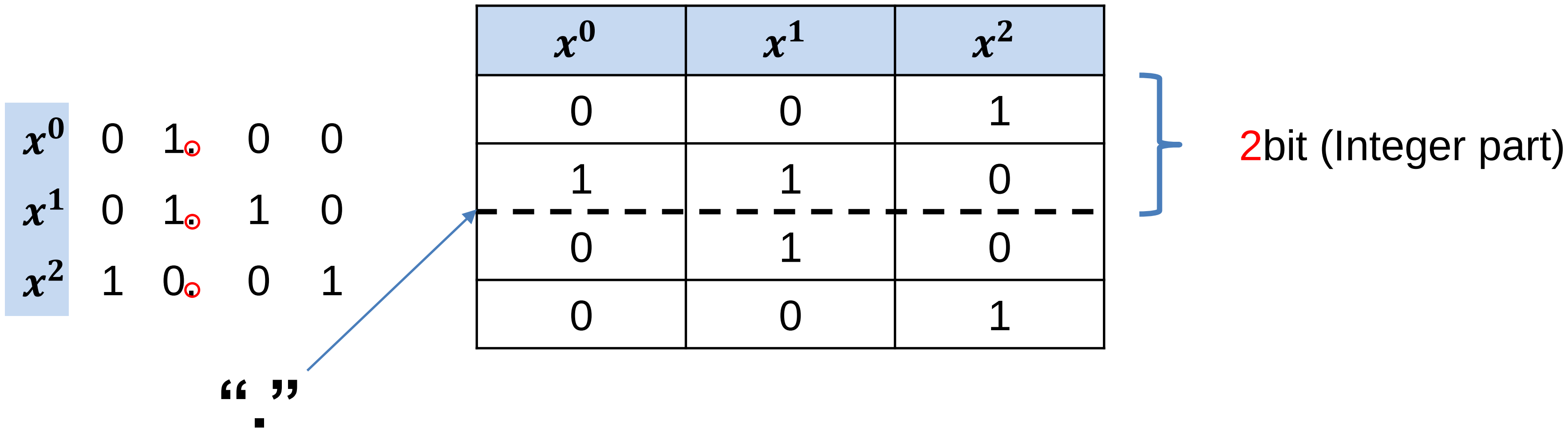
LUT

x^0	x^1	x^2	Sum
0	0	0	0
1	0	0	a_0
0	1	0	a_1
0	0	1	a_2
1	1	0	$a_0 + a_1$
1	0	1	$a_0 + a_2$
0	1	1	$a_1 + a_2$
1	1	1	$a_0 + a_1 + a_2$

$$\left(\begin{array}{c}\frac{1}{2} \times \left(\begin{array}{c}\frac{1}{2} \times \left(\begin{array}{c}\frac{1}{2} \times (a_2) \\ +(a_1)\end{array}\right) \\ +(a_0 + a_1)\end{array}\right) \\ +(a_2)\end{array}\right)$$

Correct the position of Decimal Point

$$f(x) = a_0 + a_1 x + a_2 x^2$$



x^0	0	1.	0	0
x^1	0	1.	1	0
x^2	1	0.	0	1

x^0	x^1	x^2
0	0	1
1	1	0
0	1	0
0	0	1

“.”

2bit (Integer part)

$$f(x) = \left(\frac{1}{2} \times \left(\frac{1}{2} \times \left(\frac{1}{2} \times (a_2) + (a_1) \right) + (a_0 + a_1) \right) + (a_2) \right) \times (2)^{2}$$

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Calculation Amount “Without DA”

$$f(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9$$

Direct calculation without DA

$$f(x) = a_0 + \underbrace{x^1}\left(a_1 + \underbrace{x^1}\left(a_2 + \underbrace{x^1}\left(a_3 + \underbrace{x^1}\left(a_4 + \underbrace{x^1}\left(a_5 + \underbrace{x^1}\left(a_6 + \underbrace{x^1}\left(a_7 + \underbrace{x^1}\left(a_8 + a_9\underbrace{x^1}\right)\right)\right)\right)\right)\right)\right)\right)$$

9 Multiplications

DA with 10 terms

$$\boldsymbol{f(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9}$$

Multiplication: x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9

8 times

Look Up Table

x^0	x^1	x^2	…	x^9	Output Data
0	0	0	…	0	0
1	0	0	…	0	a_0
0	1	0	…	0	a_1
…	…	…	…	…	…
1	1	1	…	1	$a_0 + a_1 + a_2 + \cdots + a_9$

2^{10} ↓ **1024**

10

DA with 5 terms × 2

$$\boldsymbol{f(x) = \underbrace{\left(a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8\right)}_{\text{DA}} + x^1\underbrace{\left(a_1 + a_3x^2 + a_5x^4 + a_7x^6 + a_9x^8\right)}_{\text{DA}}}$$

Multiplication: x^2, x^4, x^6, x^8, $x^1(a_1 + \cdots + a_9x^8)$

5 times

Look Up Table

x^0	x^2	x^4	x^6	x^8	Output Data
0	0	0	0	0	$0,\ 0$
1	0	0	0	0	$a_0,\ a_1$
0	1	0	0	0	$a_2,\ a_3$
…	…	…	…	…	…
1	1	1	1	1	$a_0+a_2+a_4+a_6+a_8,$ $a_1+a_3+a_5+a_7+a_9$

2^5 ↓ **32**

× 2

5

10 terms versus 5 terms × 2

Distributed Arithmetic
with **10 terms**

→

Distributed Arithmetic
with **5 terms × 2**

$\frac{1}{16}$ size of LUT

Almost half number of multiplications

$$f(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9$$

$$f(x) = \underline{(a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8)} + x^1\underline{(a_1 + a_3x^2 + a_5x^4 + a_7x^6 + a_9x^8)}$$

Look Up Table

x^0	x^1	x^2	…	x^9	Output
1	0	0	…	0	a_0
0	1	0	…	0	a_1
0	0	1	…	0	a_2
…	…	…	…	…	…
1	1	1	…	1	$a_0 + a_1 + a_2 + \cdots + a_9$

2^{10} ↓ **1024**

10

Multiplication: x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9

8 times

x^0	x^2	x^4	x^6	x^8	Output Data	x^0	x^2	x^4	x^6	x^8	Output Data
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	a_0	1	0	0	0	0	a_1
0	1	0	0	0	a_2	0	1	0	0	0	a_3
…	…	…	…	…	…	…	…	…	…	…	…
1	1	1	1	1	$a_0 + a_2 + a_4 + a_6 + a_8$	1	1	1	1	1	$a_1 + a_3 + a_5 + a_7 + a_9$

2^5 ↓ **32**

5

Multiplication: x^2, x^4, x^6, x^8

5 times

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Conclusion

Calculation of Taylor-series expansion

Distributed arithmetic & **Division of terms**

⇒ Reduction of LUT memory size

$1024 \;\Rightarrow\; 32 \times 2$

⇒ Reduction of x^n calculation by Multiplier

9 ⇒ 5

Q & A

- How about dividing the original formula into 3 or 4 parts?
  - As considering 10 terms formula, the number of multiplication is the fewest when it’s divided into 2 parts.

- Are you planning to try this arithmetic in real circuit?
  - I’m not sure but I want to try.

References

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Finally

Thank you all for listening.
It was a pleasure being here today.