Sampling Jitter and Finite Aperture Time Effects in Wideband Data Acquisition Systems

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SUMMARY This paper presents an explicit analysis of the output error power in wideband sampling systems with finite aperture time in the presence of sampling jitter. Sampling jitter and finite aperture time affect the ability of wideband sampling systems to capture high-frequency signals with high precision. Sampling jitter skews data acquisition timing points, which causes large errors in high-frequency (large slew rate) signal acquisition. Finite sampling-window aperture works as a low pass filter, and hence it degrades the high-frequency performance of sampling systems. In this paper, we discuss these effects explicitly not only in the case that either sampling jitter or finite aperture time exists but also the case that they exist together, for any aperture window function (whose Fourier transform exists) and sampling jitter of Gaussian distribution. These would be useful for the designer of wideband sampling data acquisition systems to know how much sampling jitter and aperture time are tolerable for a specified SNR. Some experimental measurement results as well as simulation results are provided as validation of the analytical results.

key words: jitter, phase noise, aperture time, aperture window, sampling, equivalent-time sampling, ADC, track/hold circuit, digitizing oscilloscope

1. Introduction

Digitizing oscilloscopes are widely used to capture highfrequency input signals and analyze them in the time domain [1]–[3]. In this application, sampling jitter and finite aperture time become more crucial as the input signal frequency increases; sampling jitter skews data acquisition timing points, which causes large errors in high-frequency (large slew rate) signal acquisition. Also finite sampling-window aperture, as defined in Sect. 3, works as a low-pass filter—it attenuates high-frequency components in the input signal, and significantly degrades the high-frequency performance of sampling systems. These can be serious performance-limiting fac-

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tors in wideband high-precision data acquisition systems. In this paper, we discuss these effects explicitly – not only in the case that sampling jitter and finite aperture time exist individually, but also when they exist together, for any aperture window function (which is represented by a Fourier series or whose Fourier transform exists) with Gaussian sampling jitter, and derive some formulas to represent them. These results would be useful for designing wideband sampling data acquisition systems.

Section 2 studies sampling jitter effect problems, and derives several formulas. Some experimental results as well as numerical simulation results are provided as validation of the analytical results. Section 3 discusses finite aperture time effects, and some formulas to represent them are derived. Section 4 presents the effects of both sampling jitter and finite aperture time together, and a general formula to represent them is derived. Section 5 provides our conclusions.

2. Sampling Jitter Effects

In wideband high-precision sampling systems, sampling jitter (sampling clock phase noise) is a crucial factor which affects performance [4]–[12]. Consider a sinusoidal input

$$V_{in}(t) = A\cos(2\pi f_{in}t)$$

and a sampling system which samples it at time $t = nT_s + \epsilon_n$ (Figs. 1 and 2), where T_s is the average sampling period and ϵ_n is the sampling jitter, and n = ... - 2, -1, 0, 1, 2, ...; we assume that ϵ_n follows a Gaussian distribution of $N(0, \sigma_j^2)$. Then we obtain a sampling system output of $V_{out}(nT_s) = V_{in}(nT_s + \epsilon_n)$,



Fig. 1 A sampling system, where $V_{in}(t)$ is an analog input and $V_{out}(n)$ is sampled output (n = ... - 2, -1, 0, 1, 2, ...). It is assumed that the sampling is performed during finite aperture time with an aperture window, and the sampling clock has a period of T_s with the sampling jitter ϵ_n .

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Fig. 2 (a) The sampling system in Fig. 1 samples a sinusoidal input with a sampling period of T_s . (b) Error of the sampled output due to the sampling jitter. When the sampling clock has the sampling jitter ϵ_n at *n*-th clock period, the sampled output is $V_{in}(nT_s + \epsilon_n)$ while the ideal output is $V_{in}(nT_s)$; hence the error due to the sampling jitter is given by $V_{in}(nT_s + \epsilon_n) - V_{in}(nT_s)$.

and the error between the input and output is given by

$$V_{out}(nT_s) - V_{in}(nT_s) \approx \epsilon_n \frac{dV_{in}(t)}{dt}\Big|_{t=nT}$$
(1)

where

$$2\pi f_{in}\sigma_i \ll 1 \tag{2}$$

is assumed. We see that the error is proportional to the signal slew-rate $dV_{in}(t)/dt$ as well as the sampling jitter ϵ_n . Therefore note that as the input signal frequency f_{in} increases, the sampling jitter effect becomes more serious because the signal slew-rate is proportional to f_{in} , and hence in wideband sampling systems, the sampling jitter is very problematic. M. Shinagawa et al. showed in [6] that with the assumption of Eq. (2), the error power due to the sampling jitter is given by

$$P_j = 2\pi^2 f_{in}^2 A^2 \sigma_j^2.$$
 (3)

Since the signal power of V_{in} is $A^2/2$, SNR due to the sampling jitter is given by

$$SNR = 10 \log \frac{A^2/2}{2\pi^2 f_{in}^2 A^2 \sigma_j^2} = -20 \log(2\pi f_{in} \sigma_j) \text{ [dB]}.$$
 (4)

However for the very high input frequency f_{in} , the assumption given by Eq. (2) is not satisfied and hence the above result cannot be applied to such a case. In

wideband sampling systems such as wideband digitizing oscilloscopes [1]–[3], and given σ_j , the input signal frequency f_{in} is usually too high for $2\pi f_{in}\sigma_j \ll 1$ to hold.

In this section we derive exact formulas for the error power due to the sampling jitter for almost any input signal without assuming $2\pi f_{in}\sigma_j \ll 1$, and our numerical simulations and some measurement results validate these results. Then we discuss the relationship between sampling jitter and quantization noise problems in ADC systems. These results can be useful for wideband sampling system applications.

2.1 Sampling Jitter Effects for Sinusoidal Inputs

Let us consider the sampling jitter effect for a sinusoidal input without the assumption of Eq. (2).

Time Domain Definition: We will define the error power P_j due to the sampling jitter as follows:

$$P_j \triangleq E\left[\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left[V_{out}(nT_s) - V_{in}(nT_s)\right]^2\right].$$
(5)

Here E[x] denotes the ensemble average of x, and we assume ergodicity for input and output signals.

Proposition 1: (i) The exact error power P_j due to the sampling jitter for $V_{in}(t) = A\cos(2\pi f_{in}t)$ is given by

$$P_j = A^2 \left[1 - \exp(-2\pi^2 f_{in}^2 \sigma_j^2) \right].$$
 (6)

Note that P_j depends on the input signal frequency f_{in} , but it does not depend on the sampling frequency $f_s(=1/T_s)$.

(ii) Since signal power is equal to $A^2/2$, SNR is given by

$$SNR = -10\log 2\left(1 - \exp(-2\pi^2 f_{in}^2 \sigma_j^2)\right) \text{ [dB]. (7)}$$

(iii) When $2\pi f_{in}\sigma_j \ll 1$,

$$P_j \approx A^2 \left[1 - (1 - 2\pi^2 f_{in}^2 \sigma_j^2) \right] = 2\pi^2 f_{in}^2 \sigma_j^2 A^2 \quad (8)$$

which corresponds to the result in [6].

Note: (i) Our proof of Proposition 1 is given in Appendix A.

(ii) S. Awad derived the exact formula for the error power due to the sampling jitter for a sinusoidal input without assuming Eq. (2) [7].

(iii) The definition of the error power due to the sampling jitter given in Eq. (5) is suitable for time domain applications and this may be called the *time domain definition*. However, we remark that the following frequency domain definition is widely used, such as in the ADC performance testing [12].

Frequency Domain Definition: (i) Error power (P_{fe}) due to the sampling jitter is defined as the output power of the total frequency components except for f_{in} .

(ii) Signal output power (P_{fs}) is defined as the output power of the frequency component of f_{in} .

Note that when we use this frequency domain definition, as the $f_{in}\sigma_j$ increases, the error power P_{fe} increases and the signal power P_{fs} decreases. Also

$$P_{fe} + P_{fs} = \frac{A^2}{2} \tag{9}$$

always holds. This corresponds to the fact that in average the sampling jitter does not produce or lose any power in the sampled output signal, and the average input power and the average sampled output power are equal. We note that the results in the time domain definition of error power and signal power as well as SNR due to the sampling jitter are different from those in the frequency domain definition; when we use the time domain definition, the error power is given by Eq. (6) and the signal output power is considered as $A^2/2$ regardless of f_{in} and σ_j for $V_{in}(t) = A \cos(2\pi f_{in}t)$. In this paper, however, since we target applications in time domain waveform measurement instruments such as digitizing oscilloscopes, we use the definition given by (5).

2.2 Sampling Jitter Effects for (Almost) Any Input Signal

This section describes the error power and SNR due to the sampling jitter for *any input signal* which is represented by a Fourier series or whose Fourier transform exists.

Proposition 2: (i) When the input $V_{in}(nT_s)$ is periodic with a fundamental frequency of f_0 , it can be represented by a Fourier series:

$$V_{in}(nT_s) = \frac{a_0}{2} + \sum_{k=1}^{N} \left[a_k \cos(2\pi k f_0 nT_s) + b_k \sin(2\pi k f_0 nT_s) \right].$$
(10)

In this case, the error power P_j due to the sampling jitter is given by

$$P_j = \sum_{k=1}^{N} (a_k^2 + b_k^2) \left[1 - \exp(-2\pi^2 (kf_0)^2 \sigma_j^2) \right] \quad (11)$$

Proof: See Appendix A.

(ii) P_j given by Eq. (11) is less than or equal to two times of [input signal power—its DC power]):

$$P_j \le 2 \times \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$
 (12)

In other words, when $a_0 = 0$, $P_j \leq 2 \times$ [input signal power] and hence SNR due to the sampling jitter approaches $-3 \,\mathrm{dB}$ as $2\pi f_0 \sigma_j$ increases. This can be explained as follows:

$$P_{j} = E \left[(V_{out} - V_{in})^{2} \right]$$

= $E[V_{out}^{2}] - 2E[V_{out}V_{in}] + E[V_{in}^{2}],$ (13)

where $E[V_{out}^2] = E[V_{in}^2]$ and as σ_j increases, $E[V_{out}V_{in}]$ (correlation between V_{out} and V_{in}) approaches zero, and hence P_j approaches $2 \times E[V_{in}^2]$.

This idea can be extended to any stationary input signal which is not necessarily periodic but whose Fourier transform exists $(\int_{-\infty}^{\infty} |V_{in}(t)| dt < \infty)$.

Proposition 3: (i) Suppose that the Fourier transform of $V_{in}(t)$ is $F(j\omega)$, then the error power P_j due to the sampling jitter is given by

$$P_j = \frac{1}{\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \left(1 - \exp\left(-\frac{\omega^2 \sigma_j^2}{2}\right) \right) d\omega. \quad (14)$$

Proof: See Appendix B.

(ii) According to the Parceval's theorem [14], the signal power is given by

$$S = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| F(j\omega) \right|^2 d\omega.$$
(15)

Hence SNR approaches $-3 \,\mathrm{dB}$ as σ_j increases.

Note: In general Fourier series analysis can be used for a periodic signal whose fundamental frequency is f_0 , while Fourier transform analysis can be used for stationary signals which are not necessarily periodic.

2.3 Numerical Simulation

We have performed numerical simulations to validate the above results with the following four input signals. **Example 1:** Sinusoidal input:

$$V_{in}(nT_s) = \cos(2\pi f_{in}nT_s) \tag{16}$$

Figures 3(a) and (b) show the calculated results (based on Eqs. (6) and (7)) and the simulation results for the error power and SNR due to the sampling jitter for 102, 400 samples.

Example 2: Square-wave signal approximation:

$$V_{in}(nT_s) = \frac{4}{\pi} \sum_{k=1}^{40} \left[a_k \cos(2\pi k f_0 nT_s) \right], \qquad (17)$$

where

$$a_k = \begin{cases} 1/k & (k: \text{ odd}) \\ 0 & (k: \text{ even}). \end{cases}$$

Example 3: Triangular-wave signal approximation:

$$V_{in}(nT_s) = \frac{8}{\pi^2} \sum_{k=1}^{30} \left[b_k \cos(2\pi k f_0 nT_s) \right],$$
 (18)

where

$$b_k = \begin{cases} (-1)^{(k-1)/2}/k & (k: \text{ odd}) \\ 0 & (k: \text{ even}) \end{cases}$$



Fig. 3 (a) The calculation (based on Eq. (6)) and simulation results for the error power due to the sampling jitter versus $2\pi f_{in}\sigma_j$ with a sinusoidal input of $V_{in}(t) = \cos(2\pi f_{in}t)$ for 102,400 samples. (b) The calculation (based on Eq. (7)) and simulation results for SNR due to the sampling jitter with a sinusoidal input of $V_{in}(t) = \cos(2\pi f_{in}t)$ for 102,400 samples.

Example 4: Saw-wave signal approximation:

$$V_{in}(nT_s) = \frac{2}{\pi} \sum_{k=1}^{60} \left[\frac{(-1)^{k+1}}{k} \cos(2\pi k f_0 nT_s) \right].$$
(19)

Figures 4(a), 5(a) and 6(a) illustrate the input waveforms of Examples 2, 3 and 4 respectively. Figures 4(b), 4(c), 5(b), 5(c), 6(b) and 6(c) show the calculated results (based on Eq. (11)) and the simulation results for the error power and SNR due to the sampling jitter with 102, 400 samples. We see that, for all of Examples 1, 2, 3 and 4, numerical simulations of the error power due to the sampling jitter match well with Eq. (6) or (11), and SNR approaches $-3 \,\mathrm{dB}$ as $f_{in}\sigma_j$ and $f_0\sigma_j$ increase.

2.4 Experimental Results

This subsection describes experimental results for the formula in Eq. (6) of sampling jitter effects for a sinusoidal input. Figure 7 shows the measurement



Fig. 4 (a) Waveform of $V_{in}(t)$ given by Eq. (17). (b) The calculation (based on Eq. (11)) and simulation results for the error power due to the sampling jitter versus $2\pi f_0 \sigma_j$ with a square-wave-like input of Eq. (17) for 102,400 samples. (c) The calculation (based on Eq. (11)) and simulation results for SNR due to the sampling jitter with a square-wave-like input of Eq. (17) for 102,400 samples.

setup, where we use two frequency synthesizers (Anritsu 69147B and Anritu MG3601) which are synchronized. The input signal to the sampling oscilloscope is from one synthesizer (Anritsu 69147B) whose output is frequency-modulated while its trigger signal is from the other synthesizer (Anritu MG3601) whose output is not frequency-modulated, and the carrier frequency





Fig. 5 (a) Waveform of $V_{in}(t)$ given by Eq. (18). (b) The calculation (based on Eq. (11)) and simulation results for the error power due to the sampling jitter versus $2\pi f_0 \sigma_j$ with a triangular-wave-like input of Eq. (18) for 102,400 samples. (c) The calculation (based on Eq. (11)) and simulation results for SNR due to the sampling jitter with a triangular-like input of Eq. (18) for 102,400 samples.

(center frequency) of the former synthesizer (Anritsu 69147B) output is identical to the frequency of the latter synthesizer (Anritu MG3601) output. The variation of the input waveforms due to the jitter caused by the frequency modulation is recorded on the sampling oscilloscope (HP54750A); it performs equivalent-time sampling using so-called a sequential sampling method [1]

Fig. 6 (a) Waveform of $V_{in}(t)$ given by Eq. (19). (b) The calculation (based on Eq. (11)) and simulation results for the error power due to the sampling jitter versus $2\pi f_0 \sigma_j$ with a saw-wave-like input of Eq. (19) for 102,400 samples. (c) The calculation (based on Eq. (11)) and simulation for SNR due to the sampling jitter with a saw-wave-like input of Eq. (19) for 102,400 samples.

where each sampling pulse is generated sequentially by a fine time resolution step for the corresponding trigger with the timing reference of the trigger timing point. Figure 8 shows examples of measured waveforms with the jitter, and we see that larger errors are observed as the jitter increases. Figure 9 shows an example of measured jitter histogram and acquired data, and note in Fig. 9 that the waveform recorded at the center be-



Fig. 7 Measurement setup for sampling jitter effects.



Fig. 8 Examples of measured waveforms with sampling jitter.

tween them corresponds to the reference input signal which is not frequency-modulated. We have calculated errors due to the sampling jitter by subtracting the measured data with frequency modulation from that without frequency modulation, and performed statistical calculations for them. Figure 10 shows the measured error power ("experiment" in the graph) and the calculated error power based on Eq. (6) ("new formula" in the graph and that based on Eq. (3) ("previous formula" in the graph). We see that the measured error power ("experiment") matches well with the calculated



Fig. 9 Example of jitter histogram and acquired data.



Fig. 10 Measured error power due to the sampling jitter ("experiment"), calculated error power based on Eq. (6) ("new formula") and calculated error power based on Eq. (3) ("previous formula").

error power based on Eq. (6) ("new formula") in all regions. On the other hand, the measured error power ("experiment") matches well with the calculated error power based on Eq. (3) ("previous formula") only in the region where the sampling jitter is small. Hence it is confirmed experimentally that the formula of Eq. (6) is superior to that of Eq. (3).

2.5 Sampling Jitter and Quantization Noise

An analog-to-digital converter (ADC) performs quantization operations as well as sampling operations for an analog input signal and hence it produces quantization noise [12]. In this subsection, we study the relationships between quantization noise and error power due to the sampling jitter.

Proposition 4: Letting input signal power = S, quantization noise power = Q, and

error power due to sampling jitter = P_i ,

then we obtain the SNR of the ADC as follows:

$$SNR = 10 \log \frac{S}{Q + P_j} \text{ [dB]}.$$
 (20)

This is because the quantization noise and the error

due to the sampling jitter are statistically independent, and hence their powers can be added to obtain the total noise power.

Note: (i) We have validated Proposition 4 by numerical simulation. See [11].

(ii) Calculation formula of Q for an *n*-bit ADC is wellknown [12], and hence Eq. (20) with Eq. (11) gives the SNR formula of an ADC for any input signal with respect to quantization and sampling jitter.

3. Finite Aperture Time Effects

This section describes finite aperture time effects in wideband sampling systems (Fig. 1). Ideally the sampling should be performed instantaneously at time t, however it takes some finite time during $(t - (\Delta t)/2 \sim t + (\Delta t)/2)$, and we call Δt the aperture time. Suppose that a sampling system samples the input $V_{in}(t)$ with a finite aperture time Δt (Fig. 11), and then its output is given by

$$V_{out}(t) = \int_{-\Delta t/2}^{\Delta t/2} g(\tau) V_{in}(t+\tau) d\tau$$
(21)

where $g(\tau)$ is a weighting function and we call it the aperture window [13]. Now we define the error power P_a due to the finite aperture time Δt as follows:

$$P_{a} \triangleq E \left[\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} (V_{out}(nT_{s}) - V_{in}(nTs))^{2} \right].$$
(22)

We assume here that the signal is stationary and ergodic.

Proposition 5: When the input signal is sinusoidal, given by $V_{in}(t) = A \cos(2\pi f_{in}t)$, then the error power



Fig. 11 Finite aperture time and aperture window in sampling operation.

 P_a due to the finite aperture time Δt and aperture window $g(\tau)$ is given by

$$P_{a} = \frac{A^{2}}{2} \left| 1 - G(j\omega) \right|_{\omega=2\pi f_{in}}^{2}$$
(23)

where $G(j\omega)$ is the Fourier transform of $g(\tau)$:

$$G(j\omega) \triangleq \int_{-\infty}^{\infty} g(t) \exp(-j\omega t) dt,$$
 (24)

Proof: See Appendix A.

We have considered the following three examples and performed numerical simulations of sampling systems with aperture windows. We have confirmed that the simulated error power defined in Eq. (22) and the calculated error power based on Eq. (23) match very well for all three cases; for $2\pi f_{in}\Delta t (= \omega\Delta t) = 1.5 \times 10^{-3}$, $f_{in}/f_s = 5.0 \times 10^{-4}$ and A = 1, the error power in Example 5 is 7.4×10^{-16} and that in Example 6 is 1.2×10^{-15} while that in Example 7 is 3.9×10^{-10} . These were obtained in both simulation and numerical calculation cases.

Example 5: When the aperture window $g(\tau)$ (Fig. 12(a)) and its Fourier transform $G(j\omega)$ are given by

$$g(\tau) = \begin{cases} \frac{1}{\Delta t} & \left(-\frac{\Delta t}{2} \le \tau \le \frac{\Delta t}{2}\right) \\ 0 & \text{(otherwise)} \end{cases}$$
(25)



Fig. 12 Aperture windows. (a) Example 5. (b) Example 6. (c) Example 7.

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$$G(j\omega) = \operatorname{sinc}\left(\frac{\omega\Delta t}{2}\right).$$
 (26)

Here

$$\operatorname{sinc}(x) \triangleq \frac{\sin(x)}{x}.$$
 (27)

Then the error power P_a is given by

$$P_a = \frac{A^2}{2} \left| 1 - \operatorname{sinc}\left(\frac{\omega\Delta t}{2}\right) \right|^2.$$
(28)

Example 6: When the aperture window $g(\tau)$ (Fig. 12(b)) and its Fourier transform $G(j\omega)$ are given by

$$g(\tau) = \begin{cases} \frac{2}{\Delta t} + (\frac{2}{\Delta t})^2 \tau & (-\frac{\Delta t}{2} \le \tau \le 0) \\ \frac{2}{\Delta t} - (\frac{2}{\Delta t})^2 \tau & (0 \le \tau \le \frac{\Delta t}{2}) \\ 0 & (\text{otherwise}) \end{cases}$$
(29)

$$G(j\omega) = 2\left(\frac{2}{\omega\Delta t}\right)^2 \left(1 - \cos\left(\frac{\omega\Delta t}{2}\right)\right). \tag{30}$$

Then the error power P_a is given by

$$P_a = \frac{A^2}{2} \left| 1 - 2\left(\frac{2}{\omega\Delta t}\right)^2 \left(1 - \cos\left(\frac{\omega\Delta t}{2}\right)\right) \right|^2.$$
(31)

Example 7: When the aperture window $g(\tau)$ (Fig. 12(c)) and its Fourier transform $G(j\omega)$ are given by

$$g(\tau) = \begin{cases} \frac{2}{\Delta t} \cos^2(\frac{\pi}{\Delta t}\tau) & \left(-\frac{\Delta t}{2} \le \tau \le \frac{\Delta t}{2}\right) \\ 0 & \text{(otherwise)} \end{cases}$$
(32)

$$G(j\omega) = \left[1 - \left(1 - \left(\frac{2\pi}{\omega\Delta t}\right)^2\right)^{-1}\right]\operatorname{sinc}\left(\frac{\omega\Delta t}{2}\right). \quad (33)$$

Then the error power P_a is given by

$$P_a = \frac{A^2}{2} \left| 1 - \left[1 - \left(1 - \left(\frac{2\pi}{\omega \Delta t} \right)^2 \right)^{-1} \right] \operatorname{sinc} \left(\frac{\omega \Delta t}{2} \right) \right|^2.$$
(34)

4. Combined Effects of Sampling Jitter and Finite Aperture Time

In this section we will discuss the sampling jitter and finite aperture time effects on wideband sampling systems when both exist together.

Proposition 6: (i) When the input signal is sinusoidal given by $V_{in}(t) = A\cos(2\pi f_{in}t)$, and the sampling system has the finite aperture time Δt and the aperture window $g(\tau)$ (whose Fourier transform is $G(j\omega)$) with the sampling jitter ϵ_n which follows a Gaussian distribution of $N(0, \sigma_j^2)$, then the total error power is given by as follows:

$$P_N = \frac{A^2}{2} |1 - G(j\omega)|^2 + G(\omega)_R \cdot A^2 \left(1 - \exp(-2\pi^2 f_{in}^2 \sigma_j^2)\right), \quad (35)$$

where $G(\omega)_R$ is the real part of $G(j\omega)$.

Proof: See Appendix A.

(ii) When the input $V_{in}(nT_s)$ is given by a Fourier series (Eq. (10)), then the error power P_N due to the sampling jitter and the finite aperture time is given by

$$P_{N} = \sum_{k=1}^{N} \frac{a_{k}^{2} + b_{k}^{2}}{2} \Big\{ |1 - G(j2\pi kf_{0})|^{2} + 2G_{R}(2\pi kf_{0}) \left[1 - \exp\left(-2\pi^{2}(kf_{0})^{2}\sigma_{j}^{2}\right) \right] \Big\}.$$
(36)

Proof: See Appendix A.

(iii) When the input signal is stationary and its Fourier transform $F(j\omega)$ exists, then the noise power P_N due to the sampling jitter and finite aperture time is given by

$$P_N = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \left[|1 - G(j\omega)|^2 + 2G_R(\omega) \left(1 - \exp\left(-\frac{\omega^2 \sigma_j^2}{2}\right) \right) \right] d\omega. \quad (37)$$

Proof: See Appendix B.

Note: Equations (35), (36) and (37) can be rewritten as follows:

Total error power due to

- finite aperture time and sampling jitter
- = (error power due to finite aperture time)
 - $+G_R(\omega) \times (\text{error power due to sampling jitter}).$

This can be interpreted as follows: the input signal is filtered by $G(j\omega)$ of the finite aperture function and its output amplitude is multiplied by $G_R(\omega)$. Then the error power due to the sampling jitter is given by

 $G_R(\omega) \times (\text{error power due to sampling jitter}).$

5. Conclusion

We have described the effects of sampling jitter and finite aperture time on wideband data acquisition systems, and we have explicitly derived several formulas for their effects. Numerical simulations and some experimental results have validated our results. These would be useful for the designer of wideband sampling data acquisition systems to know how much sampling jitter and aperture time are tolerable for a specified SNR.

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References

- M. Kimura, A. Minegishi, K. Kobayashi, and H. Kobayashi, "A new coherent sampling system with a triggered time interpolation," IEICE Trans. Fundamentals, vol.E84-A, no.3, pp.713–719, March 2001.
- [2] M. Kimura, K. Kobayashi, and H. Kobayashi, "A quasicoherent sampling method for wideband data acquisition," IEICE Trans. Fundamentals, vol.E85-A, no.4, April, 2002 (in press).
- [3] W.L. Gans, "The measurement and deconvolution of time jitter in equivalent-time waveform samplers," IEEE Trans. Instrumentation and Measurement, vol.32, no.1, pp.126– 133, March 1983.
- [4] A. Hajimiri and T. Lee, "A general theory of phase noise in electrical oscillators," IEEE J. Solid-State Circuits, vol.33, no.2, pp.179–193, Feb. 1998.
- B. Razavi, "A study of phase noise in CMOS oscillators," IEEE J. Solid-State Circuits, vol.31, no.3, pp.331–343, March 1996.
- [6] M. Shinagawa, Y. Akazawa, and T. Wakimoto, "Jitter analysis of high-speed sampling systems," IEEE J. Solid-State Circuits, vol.25, no.1, pp.220–224, Feb. 1990.
- [7] S.S. Awad, "Analysis of accumulated timing-jitter in the time domain," IEEE Trans. Instrumentation and Measurement, vol.47, no.1, pp.69–74, Feb. 1998.
- [8] H. Kobayashi, M. Morimura, K. Kobayashi, and Y. Onaya, "Aperture jitter effects on wideband sampling systems," Proc. IEEE Instrumentation and Measurement Technology Conference, pp.880–885, Venice, Italy, May 1999.
- [9] H. Kobayashi, M. Morimura, K. Kobayashi, and Y. Onaya, "Aperture jitter effects on wideband ADC systems," 6th IEEE International Conference on Electronics, Circuits and Systems, pp.1705–1708, Paphos, Cyprus, Sept. 1999.
- [10] H. Kobayashi, K. Kobayashi, H. Sakayori, and Y. Kimura, "ADC standard and testing in Japanese industry," Computer Standards & Interfaces, Elsevier Publishers, vol.23, pp.57–64, March 2001.
- [11] H. Kobayashi, M. Morimura, K. Kobayashi, and Y. Onaya, "Aperture jitter effects on wideband sampling systems," Proc. 12th Workshop on Circuits and Systems in Karuizawa, pp.54–59, April 1999.
- [12] B. Razavi, Principles of Data Conversion System Design, IEEE Press, 1995.
- [13] H.O. Johansson and C. Svensson, "Time resolution of NMOS sampling switches used on low-swing signals," IEEE J. Solid-State Circuits, vol.33, no.2, pp.237–245, Feb. 1998.
- [14] L. Cohen, Time-Frequency Analysis, Prentice Hall, 1995.
- [15] S. Moriguchi, K. Udagawa, and S. Ichimatsu, Mathematical Formula I, p.233, Iwanami, 1956.

Appendix A: Proof of Propositions 1, 2 (i), 5, 6 (i) and 6 (ii)

This appendix gives the proof of Propositions 1, 2 (i),

5, 6 (i) and 6 (ii). First we will consider the following case in Eq. (10), and prove Eq. (11), using Eq. (6) and the orthogonal property of sine and cosine functions.

$$V_{in}(t) \triangleq a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t),$$

= $\frac{1}{2}a_1 \left[e^{j\omega_1 t} + e^{-j\omega_1 t}\right]$
+ $\frac{1}{2}a_2 \left[e^{j\omega_2 t} + e^{-j\omega_2 t}\right]$

where $\omega_1 \triangleq 2\pi k_1 f_0$, $\omega_2 \triangleq 2\pi k_2 f_0$, and $k_1 \neq k_2$. Then if both the sampling jitter and the finite aperture time exist, the output is given as follows:

$$\begin{split} V_{out}(nT_s) \\ &= \int_{-\infty}^{\infty} g(\tau) \Big[a_1 \cos \left(\omega_1 (nT_s + \tau + \epsilon_n) \right) \\ &+ a_2 \cos \left(\omega_2 (nT_s + \tau + \epsilon_n) \right) \Big] d\tau \\ &= \frac{1}{2} \int_{-\infty}^{\infty} g(\tau) \Big(a_1 \left[e^{j\omega_1 (nT_s + \tau + \epsilon_n))} + e^{-j\omega_1 (nT_s + \tau + \epsilon_n)} \right] \\ &+ a_2 \left[e^{j\omega_2 (nT_s + \tau + \epsilon_n)} + e^{-j\omega_2 (nT_s + \tau + \epsilon_n)} \right] \Big) d\tau \\ &= \frac{a_1}{2} \left[G(j\omega_1) e^{j\omega_1 (nT_s + \epsilon_n)} + G(-j\omega_1) e^{-j\omega_1 (nT_s + \epsilon_n)} \right] \\ &+ \frac{a_2}{2} \left[G(j\omega_2) e^{j\omega_2 (nT_s + \epsilon_n)} \\ &+ G(-j\omega_2) e^{-j\omega_2 (nT_s + \epsilon_n)} \right] . \end{split}$$

Then we obtain the following:

$$(V_{out}(nT_s) - V_{in}(nT_s))^2 = P_1(n) + P_2(n) + P_3(n),$$

where

$$P_{1}(n) \triangleq \frac{a_{1}^{2}}{4} \Big[e^{j2\omega_{1}nT_{s}} \left\{ G(j\omega_{1})e^{j\omega_{1}\epsilon_{n}} - 1 \right\}^{2} \\ + e^{-j2\omega_{1}nT_{s}} \left\{ G(-j\omega_{1})e^{-j\omega_{1}\epsilon_{n}} - 1 \right\}^{2} \\ + 2 \left| G(j\omega_{1})e^{j\omega_{1}\epsilon_{n}} - 1 \right|^{2} \Big],$$

$$P_{2}(n) \triangleq \frac{a_{2}^{2}}{4} \Big[e^{j2\omega_{2}nT_{s}} \left\{ G(j\omega_{2})e^{j\omega_{2}\epsilon_{n}} - 1 \right\}^{2} \\ + e^{-j2\omega_{2}nT_{s}} \left\{ G(-j\omega_{2})e^{-j\omega_{2}\epsilon_{n}} - 1 \right\}^{2} \\ + 2 \left| G(j\omega_{2})e^{j\omega_{2}\epsilon_{n}} - 1 \right|^{2} \Big],$$

$$\begin{split} P_{3}(n) &\triangleq \frac{a_{1}a_{2}}{4} \\ &\times \left[e^{j(\omega_{1}+\omega_{2})nT_{s}} \{G(j\omega_{1})e^{j\omega_{1}\epsilon_{n}}-1\} \right. \\ &\cdot \{G(j\omega_{2})e^{j\omega_{2}\epsilon_{n}}-1\} + e^{j(\omega_{1}-\omega_{2})nT_{s}} \\ &\cdot \{G(j\omega_{1})e^{j\omega_{1}\epsilon_{n}}-1\} \{G(-j\omega_{2})e^{-j\omega_{2}\epsilon_{n}}-1\} \\ &+ e^{j(-\omega_{1}+\omega_{2})nT_{s}} \{G(-j\omega_{1})e^{-j\omega_{1}\epsilon_{n}}-1\} \\ &\cdot \{G(j\omega_{2})e^{j\omega_{2}\epsilon_{n}}-1\} + e^{-j(\omega_{1}+\omega_{2})nT_{s}} \\ &\cdot \{G(-j\omega_{1})e^{-j\omega_{1}\epsilon_{n}}-1\} \{G(-j\omega_{2})e^{-j\omega_{2}\epsilon_{n}}-1\} \Big] \end{split}$$

Then the total error power due to the sampling jitter and the finite aperture time is give by

$$P_{N} \triangleq E\left[\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(V_{out}(nT_{s}) - V_{in}(nT_{s})\right)^{2}\right]$$

= $E\left[\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(P_{1}(n) + P_{2}(n) + P_{3}(n)\right)\right].$

Here

$$\begin{split} E & \left[\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} P_1(n) \right] \\ &= \frac{a_1^2}{2} E \left[\left| G(j\omega_1) e^{j\omega_1 \epsilon_n} - 1 \right|^2 \right] \\ &= \frac{a_1^2}{2} \left[\left| 1 - G(j\omega_1) \right|^2 + 2G_R(\omega_1) (1 - e^{-\omega_1^2 \sigma_j^2/2}) \right]. \end{split}$$

$$E\left[\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} P_2(n)\right]$$

= $\frac{a_2^2}{2} E\left[\left|G(j\omega_2)e^{j\omega_2\epsilon_n} - 1\right|^2\right]$
= $\frac{a_2^2}{2}\left[\left|1 - G(j\omega_2)\right|^2 + 2G_R(\omega_2)(1 - e^{-\omega_2^2\sigma_j^2/2})\right].$

$$E\left[\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}P_3(n)\right]=0.$$

Therefore

$$P_N = \frac{a_1^2}{2} \left[|1 - G(j\omega_1)|^2 + 2G_R(\omega_1)(1 - e^{-\omega_1^2 \sigma_j^2/2}) \right] + \frac{a_2^2}{2} \left[|1 - G(j\omega_2)|^2 + 2G_R(\omega_2)(1 - e^{-\omega_2^2 \sigma_j^2/2}) \right].$$

Here $G_R(\omega)$ is the real part of $G(j\omega)$. Note that ϵ_n follows a Gaussian distribution $N(0, \sigma_j)$ and in the above proof, the followings are used [15]

$$E\left[\cos(\omega\epsilon_n)\right] = \exp\left(-\frac{\omega^2\sigma_j^2}{2}\right) \tag{A.1}$$

$$E\left[\sin(\omega\epsilon_n)\right] = 0 \tag{A.2}$$

as well as Lemma 1. We note that extension of the proof to a general N (i.e., Propositions 6 (ii)) is straightforward. Also Proposition 1 can be proved by letting $g(t) = \delta(t)$ and $a_2 = 0$, and Proposition 2 (i) can be proved by letting $g(t) = \delta(t)$ while Proposition 5 can be proved by letting $\sigma_j = 0$.

Lemma 1: When
$$\alpha \neq \beta$$
, as $T \to \infty$,

(i)
$$\frac{1}{T} \int_0^T \cos(\alpha t) \cos(\beta t) dt \to 0,$$

(ii) $\frac{1}{T} \int_0^T \sin(\alpha t) \sin(\beta t) dt \to 0,$
(iii) $\frac{1}{T} \int_0^T \cos(\alpha t) \sin(\beta t) dt \to 0.$

Proof of Lemma 1: We will prove Lemma 1 (i) here, and Lemma 1 (ii) and (iii) will be proved in a similar manner. When $\alpha \neq \beta$,

$$\frac{1}{T} \int_0^T \cos(\alpha t) \cos(\beta t) dt$$

= $\frac{1}{2T} \int_0^T \{\cos((\alpha + \beta)t) + \cos((\alpha - \beta)t)\} dt$
= $\frac{1}{2} \left[\frac{\sin((\alpha + \beta)T)}{(\alpha + \beta)T} + \frac{\sin((\alpha - \beta)T)}{(\alpha - \beta)T} \right] \rightarrow 0,$
as $T \rightarrow \infty.$

Appendix B: Proof of Propositions 3 (i) and 6 (iii)

This appendix gives the proof of Propositions 3 (i) and 6 (iii). When the Fourier transform $F(j\omega)$ of $V_{in}(t)$ exists,

$$F(j\omega) \triangleq \int_{-\infty}^{\infty} V_{in}(t) e^{-j\omega t} dt,$$

then

$$V_{in}(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega nT_s} d\omega.$$

If both the sampling jitter and the finite aperture time exist, the output $V_{out}(nT_s)$ is given by

$$\begin{split} V_{out}(nT_s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau) F(j\omega) e^{j\omega(nT_s + \tau + \epsilon_n)} d\omega d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(-j\omega) F(j\omega) e^{j\omega(nT_s + \epsilon_n)} d\omega. \end{split}$$

Then the total error power to the sampling jitter and finite aperture time is given by follows:

$$E\left[\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}(V_{out}(nT_s)-V_{in}(nT_s))^2\right]$$

= $\frac{1}{2\pi}E\left[\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\left\{\int_{-\infty}^{\infty}F(j\omega)\right\}$
 $\times e^{j\omega nT_s}[G(j\omega)e^{j\omega\epsilon_n}-1]d\omega\right\}^2\right]$
= $\frac{1}{(2\pi)^2}E\left[\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(j\omega)F(-j\omega')\right]$
 $\times [G(-j\omega)e^{j\omega\epsilon_n}-1][G(-j\omega')e^{j\omega'\epsilon_n}-1]^*d\omega d\omega'$

$$= \frac{1}{2\pi} E \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(j\omega) F(-j\omega') \times [G(-j\omega)e^{j\omega\epsilon_n} - 1] [G(-j\omega')e^{-j\omega'\epsilon_n} - 1]^* \times [G(-j\omega)e^{j\omega\epsilon_n} - 1] [G(-j\omega')e^{-j\omega'\epsilon_n} - 1]^* \times \frac{1}{2\pi} \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{j(\omega-\omega')nT_s} \right] d\omega d\omega'$$

$$= \frac{1}{2\pi} E \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(j\omega) F(j\omega')^* \times [G(j\omega)e^{j\omega\epsilon_n} - 1] [G(-j\omega')e^{-j\omega'\epsilon_n} - 1]^* \times \delta(\omega - \omega') \right] d\omega d\omega'$$

$$= \frac{1}{2\pi} E \left[\int_{-\infty}^{\infty} |F(j\omega)|^2 \times [G(j\omega)e^{j\omega\epsilon_n} - 1] [G(-j\omega)e^{-j\omega\epsilon_n} - 1]^* \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \times \left[|G(j\omega)|^2 + 1 - 2G_R(\omega)E [\cos(\omega\epsilon_n)] \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \times \left\{ |1 - G(j\omega)|^2 + 2G_R(\omega) \left[1 - \exp\left(-\frac{\omega^2 \sigma_j^2}{2}\right) \right] \right\} d\omega. \quad (A \cdot 3)$$

Here $G_R(\omega)$ is the real part of $G(j\omega)$, and in the above proof,

$$\frac{1}{2\pi} \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{j(\omega - \omega')nT_s} = \delta(\omega - \omega')$$

as well as Eqs. $(A \cdot 1)$ and $(A \cdot 2)$ are used [14], [15]. Thus Proposition 5 (iii) has been proved.

To prove Proposition 3(i), let $g(t) = \delta(t)$, and then $G(j\omega) = 1$ and hence Eq. (A·3) yields to

$$P_j = \frac{1}{\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \left(1 - \exp\left(-\frac{\omega^2 \sigma_j^2}{2}\right)\right) d\omega.$$

Then Proposition 3 (i) has been proved.



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